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Quantum field theory and Hopf algebra cohomology

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Abstract
We exhibit a Hopf superalgebra structure of the algebra of field operators of quantum field theory (QFT) with the normal product. Based on this we construct the operator product and the time-ordered product as a twist deformation in the sense of Drinfeld. Our approach yields formulae for (perturbative) products and expectation values that allow for a significant enhancement in computational efficiency as compared to traditional methods. Employing Hopf algebra cohomology sheds new light on the structure of QFT and allows the extension to interacting (not necessarily perturbative) QFT. We give a reconstruction theorem for time-ordered products in the spirit of Streater and Wightman and recover the distinction between free and interacting theory from a property of the underlying cocycle. We also demonstrate how non-trivial vacua are described in our approach solving a problem in quantum chemistry.

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Contents

1. Introduction

The purpose of this paper is to present a new approach to the algebraic and combinatorial structures at the heart of quantum field theory (QFT). This approach has merits on the practical as well as on the conceptual side. On the practical side, it allows for a major computational enhancement based on an efficient description of the combinatorics and on non-recursive closed formulae. On the conceptual side, it gives new insights into the algebraic structure of QFT. We evidence this through applications to non-perturbative QFT and non-trivial vacua.

The starting point of our approach is the identification of a Hopf algebraic structure at the core of QFT. That is, the algebra of field operators with the normal product is a *Hopf superalgebra*. This means that besides the product there is a *coproduct* that describes, intuitively speaking, the different ways in which a product of field operators might be partitioned into two sets. Indeed, it is this coproduct that plays the key role in a closed description of combinatorial structures and that allows for computationally efficient algorithms. Another key structure of the Hopf superalgebra is the *counit*. This turns out to describe the standard vacuum expectation value. Algebraically, this Hopf superalgebra is the graded symmetric Hopf algebra (described in detail in appendix A.3, see also [1, appendix 2]). The conceptual origin of this Hopf superalgebra is rather simple. Identifying the normal ordered products with functionals on field configurations, the coproduct is induced by the linear addition of fields.

The second main step consists in identifying the standard canonical quantization with a twist in the sense of Drinfeld [2]. More precisely, the operator product emerges as a twist deformation of the normal product. As is common we deal here at first with the free QFT. The twist is induced by a *Laplace pairing* which in turn is determined by a suitable propagator. Furthermore, the time-ordered product can be obtained similarly as a direct twist deformation of the normal product. In this case, the Laplace pairing is determined by the Feynman propagator. Since vacuum expectation values of time-ordered products are the main ingredients of physical scattering amplitudes this allows the use of our methods in actual calculations of physical quantities.

It is one of the basic facts in quantum field theory that Wick's theorem relates normal and time-ordered correlation functions [3–5]. It was only recently noted by Fauser that this transformation can be advantageously described in Hopf algebraic terms [6–8]. This is indeed a crucial ingredient of our construction, which allows us to prove that the twists yield the desired products. In particular, we show that the Hopf algebraic Wick transformation can be applied to yield the operator product from the normal product in a way analogous as for the time-ordered product.

Other aspects of our approach described so far are also present in the literature in different guises. Oeckl used the (dual of the) present Hopf algebra structure to generalize QFT [9, 10] to quantum group symmetries and the twist to describe QFT on noncommutative spaces [11]. Borcherds defined vertex algebras with quantum group methods [12, 13]. In particular, he uses a construction similar to the twist in a related context. The research of the present paper was initiated by Brouder [14] and a preliminary presentation of parts of this work was given in [15].

The closed formulae emerging in our framework facilitate highly efficient computations of products and expectation values. This is mainly due to the heavy use of the coproduct structure. Indeed, this is well known in combinatorics where Hopf algebraic methods are established for this purpose. We know from computer algebra calculations that precisely the techniques employed enhance performance and that in a well defined sense no algorithm can come up with fewer terms, see [16, 17]. While the twisted products described so far are the products of the free QFT our framework is naturally compatible with the usual perturbation theory and thus applicable to it. This implies that the computational advantages directly apply to perturbative QFT.

The third step consists in exploiting the Hopf algebra cohomology theory due to Sweedler, which underlies the twisted product [18]. (In that work twisted products of the type used here were also introduced for the first time.) Besides affording conceptual insight this yields immediate practical benefits. Among these is the realization of the time-ordering prescription of QFT as an algebra isomorphism. This in turn can be used on the computational side.

The cohomological point of view affords a further extension of our framework. Significantly, twisted products cannot only be defined with Laplace pairings, but with 2-cocycles, of which Laplace pairings are only a special case. Remarkably, it turns out that 2-cocycles that are not Laplace pairings lead to (non-perturbatively described) interacting QFTs. Moreover, any QFT (with polynomial fields) can be obtained in this way. A QFT is free if and only if the 2-cocycle is a Laplace pairing.

A further application of the cohomology that we develop is to non-trivial vacua. We show that changing the choice of vacuum can also be encoded through a twist. Indeed, it turns out that there is a 'duality' or correspondence between the choice of vacuum and that of product. We exemplify this result by solving a problem posed by Kutzelnigg and Mukherjee [19] regarding 'adapted normal products' in quantum chemistry. While they were able to give only examples for low orders, our framework yields closed formulae for all orders. Our method is capable of describing condensates too, as we know from [20].

Although we do not develop this point of view in the present paper, a twist in the sense used here is automatically an (equivariant) deformation quantization. Indeed, this was one of the original motivations for Drinfeld to introduce this concept [21]. This means that our approach is thus inherently connected to the deformation quantization approach to QFT. This approach starts also with the normal ordered product and views the other products as deformations of it. See the recent paper by Hirshfeld and Henselder [22].

The paper is roughly divided into three parts. The first, consisting of section 2, starts by introducing a few essential mathematical concepts. Then, the Hopf algebra structure of the normal ordered field operators is developed. Next, the operator and time-ordered products are constructed as twisted products induced by Laplace pairings. We finish the section with closed formulae for Wick expansions, the various products and expectation values showing the practical efficiency of the framework. This part of the paper is intended for a broad audience and should be readable without prior familiarity with Hopf algebras.

The second part of the paper consists of section 3. Here we go deeper into the underlying mathematics, starting with a brief review of Hopf algebra cohomology and Drinfeld twisting. Then we turn to the implications for QFT. In particular, we describe the cohomological understanding of the operator and time-ordered product as twisted products. Among other practical consequences we derive the time-ordering operation as an algebra isomorphism.

The third part of the paper presents further results emerging from the cohomological insights. It consists of sections 4 and 5. In section 4 interactions are treated. Firstly, we show that our framework is compatible with perturbation theory and thus allows the application of our methods there. Secondly, we consider general (and not necessarily perturbative) quantum field theory and show that our framework naturally extends to it. In particular, we present the reconstruction theorem that allows us to describe any (linear and polynomial) QFT through a 2-cocycle. In section 5, we show how choosing non-trivial vacua can be naturally expressed in our framework. Moreover, we demonstrate the efficient solution of a problem arising in quantum chemistry with our approach.

After conclusions and outlook the paper ends with two appendices. Appendix A gives some elementary definitions on Hopf superalgebras and in particular the graded symmetric Hopf superalgebra that plays the crucial role in this paper. The terminology of Hopf $*$-superalgebras is not unique and even in general incompatible between different sources. So a further value of this appendix is that it collects in a coherent and compatible way notions scattered in the literature. Appendix B consists firstly of a short description of the cohomology groups in the bosonic case and secondly of the more technical proofs of propositions and lemmas appearing in the main text.

We refer readers who wish to know more about Hopf algebras and quantum group theory to [23] and [24]. The latter reference is particularly suitable for the cohomology theory and the twist construction.

2. Free quantum field theory

2.1. Mathematical basis

We start in this section by introducing a few mathematical concepts that will be required throughout the paper. These are, apart from Hopf (super)algebras, the Laplace pairing and the twisted product. It should be possible even for the reader without previous experience with Hopf algebras to follow the main steps of section 2. Indeed, a first reading should be possible without paying too much attention to the details of definitions.

2.1.1. Hopf $*$-superalgebra. Recall that a *Hopf algebra* H, besides being an associative algebra with a unit, has a coassociative *coproduct* $\Delta : H \to H \otimes H$, a *counit* $\varepsilon : H \to \mathbb{C}$ and an *antipode* $\gamma : H \to H$, satisfying compatibility conditions. By definition the coproduct of an element a of H can always be written as a (non-unique) linear combination of the form $\Delta a = \sum_i a'_i \otimes a''_i$. In order to avoid the proliferation of indices it is customary to use *Sweedler's notation* for this, i.e. we write $\Delta a = \sum a_{(1)} \otimes a_{(2)}$. Due to the coassociativity Sweedler's notation extends unambiguously to multiple coproducts as follows: $\sum (a_{(1)})_{(1)} \otimes (a_{(1)})_{(2)} \otimes a_{(2)} = \sum a_{(1)} \otimes (a_{(2)})_{(1)} \otimes (a_{(2)})_{(2)} = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$ etc.

A superalgebra A is a $\mathbb{Z}_2$-graded algebra so that $|ab| = |a| + |b|$ modulo 2 where $|a|$ denotes the parity of the element a, $|a| = 0$ if it is even (bosonic) and $|a| = 1$ if it is odd (fermionic). A *Hopf superalgebra* is a superalgebra with unit and $\mathbb{Z}_2$-graded coproduct, counit and antipode. A *$*$-(super)algebra* is a (super)algebra A with an antilinear map $* : A \to A$

such that $(ab)^* = b^*a^*$. A Hopf $*$-superalgebra is a Hopf superalgebra and $*$-superalgebra where the $*$-structure is compatible with coproduct, counit and antipode.

We refer the reader who is not familiar with these notions or wishes to recall their details to appendix A, where the complete definitions are given on an elementary level.

2.1.2. Laplace pairing. We introduce now the concept of Laplace pairing which is relevant to interpret Wick's theorem in terms of Hopf superalgebras.

Let H be a Hopf superalgebra that we require to be graded cocommutative. That is, the coproduct satisfies $a_{(1)} \otimes a_{(2)} = (-1)^{|a_{(1)}||a_{(2)}|} a_{(2)} \otimes a_{(1)}$. A *pairing* on H is a linear map $(\cdot|\cdot) : H \otimes H \to \mathbb{C}$. It is called even if $(a|b) = 0$ when the parities of a and b are different. A *Laplace pairing*[5] on H is an even pairing on H such that the product and the coproduct are dual in the sense that

$$(aa'|b) = \sum (-1)^{|a'||b_{(1)}|} (a|b_{(1)})(a'|b_{(2)}) \tag{1}$$

$$(a|bb') = \sum (-1)^{|a_{(2)}||b|} (a_{(1)}|b)(a_{(2)}|b') \tag{2}$$

and the unit and counit are dual as follows:

$$(1|a) = (a|1) = \varepsilon(a). \tag{3}$$

2.1.3. Twisted product. A Laplace pairing on H can be used to deform the product of H. Sweedler [18] defined the twisted product on H, which we denote by $\circ$, as[6]

$$a \circ b = \sum (-1)^{|a_{(2)}||b_{(1)}|} (a_{(1)}|b_{(1)}) a_{(2)} b_{(2)}. \tag{4}$$

The twisted product $\circ$ is associative, and 1 is also the unit for $\circ$. As we shall see later, this twisted product yields an elegant way to write Wick's theorem. From the mathematical point of view, this arbitrary seeming definition can be understood as a special case of the more fundamental Drinfeld twist discussed in section 3.2.

Note also that the twisted product is not compatible with the coproduct: in general $\Delta(a \circ b) \neq \sum (-1)^{|b_{(1)}||a_{(2)}|} (a_{(1)} \circ b_{(1)}) \otimes (a_{(2)} \circ b_{(2)})$.

2.2. The Hopf superalgebra of creation and annihilation operators

In this section, we define the Hopf superalgebra of creation and annihilation operators. The fermion and boson creation and annihilation operators will be treated in a unified way. Normal products of creation and annihilation operators form a well-known graded commutative superalgebra, called the symmetric superalgebra (see appendix A). In this section, we define a coproduct and a counit which are compatible with the normal product, and we equip the normal products of operators with the structure of a Hopf superalgebra. The Hopf superalgebra of creation and annihilation operators will be used to define the Hopf superalgebra of quantum fields.

We first denote by $\varphi(x; s)$ the solutions of the classical field equations (e.g. the classical Klein–Gordon, Dirac or Maxwell equations). The solution $\varphi(x; s)$ is a function of the

[5] This name was given by Rota some time ago [25, 26], because equations (1) and (2) are an elegant way of writing the Laplace identities for determinants. Equation (1) is called expansion by rows and equation (2) expansion by columns. They express the determinant in terms of minors (see [27], p 26 and [28], p 93) and were derived by Laplace in 1772 [29].

[6] Sweedler considered only the bosonic case. He called the product a crossed product and his definition was somewhat more general (in a different direction).

spacetime variable x and s indexes the solutions of the classical field equations. In the vacuum, s is the 3-momentum $\mathbf{k}$ for scalar fields. For Dirac fields there are positive energy solutions $\varphi_>(x, s)$ with $s = (\mathbf{k}, \alpha)$ and $\alpha = 1, 2$ for up and down spin states and negative energy solutions $\varphi_<(x, s)$ with $s = (\mathbf{k}, \alpha)$ and $\alpha = 1, 2$ for up and down spin states. For the classical field equations in an external potential, s is a discrete index for the bound states and a continuous index for the scattering states. For $x = (0, \mathbf{r})$, the set of functions $\varphi(x; s)$ is assumed to form a suitable space of functions of $\mathbf{r}$. The functions $\varphi(x; s)$ will be used to define quantum fields.

2.2.1. The superalgebra structure. The creation and annihilation operators are denoted by $a^\dagger(s)$ and $a(s)$, respectively. They create and annihilate a particle in the state s. They are operators acting on a Fock space $\mathcal{F}$ and their precise definition is given in [30], p 218. The normalized state of $\mathcal{F}$ corresponding to no particles is called the vacuum and denoted by $|0\rangle$. The parity of these operators is $|a(s)| = |a^\dagger(s)| = 1$ for a fermion field and $|a(s)| = |a^\dagger(s)| = 0$ for a boson field. The operator product of two operators u and v is written uv. The (anti)commutation relations among creation operators and among annihilation operators of bosons (fermions) [31] can be summarized as

$$\begin{aligned} a(s)a(s') &= (-1)^{|a(s)||a(s')|}a(s')a(s) \\ a^\dagger(s)a^\dagger(s') &= (-1)^{|a^\dagger(s)||a^\dagger(s')|}a^\dagger(s')a^\dagger(s). \end{aligned} \tag{5}$$

These equations mean that two annihilation operators or two creation operators commute for bosons, anticommute for fermions and commute for a boson and a fermion.

The superalgebra $\hat{A}_N$ of normal products is generated as a vector space by products of creation operators on the left of products of annihilation operators. For example $u = a^\dagger(s_1)\cdots a^\dagger(s_m)a(s_{m+1})\cdots a(s_{m+n})$ is an element of $\hat{A}_N$. The parity of u is $|u| = |a^\dagger(s_1)| + \cdots + |a^\dagger(s_m)| + |a(s_{m+1})| + \cdots + |a(s_{m+n})|$ modulo 2. The element given by $m = 0$ and $n = 0$ in this example is the unit operator denoted by 1. The product in $\hat{A}_N$ is the normal product. In quantum field theory, the normal product of two elements u and v is denoted by $:uv:$. Since this notation becomes cumbersome when we manipulate various products of several fields, we prefer to denote the normal product by $u \vee v$, which is the standard mathematical notation for a graded-commutative product. The normal product is defined by $a^\dagger(s) \vee a^\dagger(s') = a^\dagger(s)a^\dagger(s')$, $a^\dagger(s) \vee a(s') = a^\dagger(s)a(s')$, $a(s) \vee a^\dagger(s') = (-1)^{|a(s)||a^\dagger(s')|}a^\dagger(s')a(s)$, $a(s) \vee a(s') = a(s)a(s')$ and extended to $\hat{A}_N$ by associativity and linearity. From the definition of the normal product and the relations (5), we see that if u and v are in $\hat{A}_N$, $u \vee v = (-1)^{|u||v|} v \vee u$. That is, the normal product is graded commutative. Hence, the superalgebra $\hat{A}_N$ is a graded-commutative associative superalgebra with unit 1. These results can be summarized in the following proposition:

Proposition 2.1. *If $\hat{V}$ is the vector space generated by $a(s)$ and $a^\dagger(s)$ (for all s), the superalgebra $\hat{A}_N$ of normal products has the structure of the symmetric superalgebra $\mathsf{Sym}(\hat{V})$.*

The symmetric superalgebra $\mathsf{Sym}(\hat{V})$ is described in appendix A. If the theory contains bosons and fermions the vector space $\hat{V}$ generated by $a(s)$ and $a^\dagger(s)$ for all s can be written as $\hat{V} = \hat{V}_0 \oplus \hat{V}_1$, where $\hat{V}_0$ is generated by the boson operators and $\hat{V}_1$ by the fermion operators.

From appendix A, we know that $\mathsf{Sym}(\hat{V})$ has the structure of a Hopf superalgebra. Thus, the superalgebra of creation and annihilation operators has a Hopf superalgebra structure that will be discussed in the following section. For later convenience, we distinguish the superalgebra $\hat{A}_N$ of creation and annihilation operators without the full Hopf structure, and the Hopf superalgebra of creation and annihilation operators, that we denote by $\hat{H}$.

2.2.2. The Hopf $*$-superalgebra structure. Starting from the Hopf superalgebra $\mathsf{Sym}(\hat{V})$, we see that the coproduct of the Hopf superalgebra $\hat{H}$ of creation and annihilation operators is defined by $\Delta 1 = 1 \otimes 1$, $\Delta a(s) = a(s) \otimes 1 + 1 \otimes a(s)$, $\Delta a^\dagger(s) = a^\dagger(s) \otimes 1 + 1 \otimes a^\dagger(s)$ on $\hat{V}$ and extended to $\hat{H}$ by $\Delta(u \vee v) = \sum (-1)^{|u_{(2)}||v_{(1)}|}(u_{(1)} \vee v_{(1)}) \otimes (u_{(2)} \vee v_{(2)})$. For example, if $a = a(s)$, $b = a(s')$ and $c = a(s'')$,

$$\Delta(a \vee b) = (a \vee b) \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a + 1 \otimes (a \vee b)$$

$$\begin{aligned}\Delta(a \vee b \vee c) = 1 \otimes a \vee b \vee c + a \otimes b \vee c + (-1)^{|a||b|} b \otimes a \vee c + (-1)^{|a||c|+|b||c|} c \otimes a \vee b \\ + a \vee b \otimes c + (-1)^{|b||c|} a \vee c \otimes b + (-1)^{|a||b|+|a||c|} b \vee c \otimes a + a \vee b \vee c \otimes 1.\end{aligned}$$

In general

$$\Delta(u^1 \vee \cdots \vee u^n) = \sum (-1)^F u^1_{(1)} \vee \cdots \vee u^n_{(1)} \otimes u^1_{(2)} \vee \cdots \vee u^n_{(2)}$$

for any $u^1, \ldots, u^n \in \hat{H}$ and with $F = \sum_{k=2}^{n} \sum_{l=1}^{k-1} |u^k_{(1)}||u^l_{(2)}|$.

In particular, if $a_1, \ldots, a_n$ are creation or annihilation operators, the coproduct of $a_1 \vee \cdots \vee a_n$ is given by equation (36) of appendix A.3.

The counit of $\hat{H}$ is defined by $\varepsilon(1) = 1, \varepsilon(a(s)) = 0$ and $\varepsilon(a^\dagger(s)) = 0$ and extended to $\hat{H}$ by $\varepsilon(u \vee v) = \varepsilon(u)\varepsilon(v)$ for any u and v in $\hat{H}$. Therefore, $\varepsilon(u) = 0$ if $u = a^\dagger(s_1) \cdots a^\dagger(s_m) a(s_{m+1}) \cdots a(s_{m+n})$ for $m > 0$ or $n > 0$. The relation between Hopf algebra and quantum field concepts is strengthened by the following:

Proposition 2.2. *For any normal product, i.e. any element $u \in \hat{H}$, the counit is equal to the vacuum expectation value: $\varepsilon(u) = \langle 0|u|0\rangle$.*

To show this, we evaluate $\epsilon(u)$ and $\langle 0|u|0\rangle$ for all elements of a basis of $\hat{H}$. The proposition is true for the unit because $\varepsilon(1) = 1 = \langle 0|1|0\rangle$. Take now a basis element of $\hat{H} u = a^\dagger(s_1) \cdots a^\dagger(s_m) a(s_{m+1}) \cdots a(s_{m+n})$ for $m > 0$ or $n > 0$. Then $\varepsilon(u) = 0 = \langle 0|u|0\rangle$. The result follows for all elements of $\hat{H}$ by linearity of the counit and of the vacuum expectation value. This relation between the counit and the expectation value over the vacuum was already pointed out in [6].

To complete the description of the Hopf superalgebra $\hat{H}$, we define its antipode by $\gamma(a^\#(s_1) \vee \cdots \vee a^\#(s_n)) = (-1)^n a^\#(s_1) \vee \cdots \vee a^\#(s_n)$, where $a^\#(s_i)$ stands for $a^\dagger(s_i)$ or $a(s_i)$. Moreover, $\hat{H}$ has a $*$-structure generated by $a(s)^* = a^\dagger(s)$.

2.3. The Hopf superalgebra of field operators

The operators used in the superalgebra $\hat{A}_N$ of normal products are independent of space and time, they are indexed by the solutions of the classical equation. Now we introduce space- and time-dependent field operators for Dirac and scalar fields. An excellent description of field operators can be found in [30].

2.3.1. The field operators. To define the Dirac field operator, we need to split the set of solutions of the Dirac equation into two groups, the positive energy states $\varphi_>(x; s)$ and the negative energy states $\varphi_<(x, s)$. The solutions with positive energy are $\varphi_>(x; n)$ with energy $E_n < m$ (where m is the electron mass) for bound states and $\varphi_>(x; \mathbf{k}, \alpha)$ with energy $\omega_k = \sqrt{\mathbf{k} \cdot \mathbf{k} + m^2}$ for continuum states. The solutions with negative energy are assumed to be always continuum states $\varphi_<(x; \mathbf{k}, \alpha)$ with energy $-\omega_k$.

The Dirac field operator is defined by [31]

$$\psi(x) = \sum_n \varphi_>(x; n) b_n + \int \mathrm{d}\mu(\mathbf{k}) \sum_{\alpha=1}^{2} \varphi_>(x; \mathbf{k}, \alpha) b_\alpha(\mathbf{k}) + \varphi_<(x; \mathbf{k}, \alpha) d^\dagger_\alpha(\mathbf{k}).$$

In this expression, $\mathrm{d}\mu(\mathbf{k}) = m/(8\pi^3\omega_k)\mathrm{d}\mathbf{k}$. Its Dirac adjoint is

$$\overline{\psi}(x) = \sum_n \bar{\varphi}_>(x;n)b_n^\dagger + \int \mathrm{d}\mu(\mathbf{k}) \sum_{\alpha=1}^{2} \bar{\varphi}_>(x;\mathbf{k},\alpha)b_\alpha^\dagger(\mathbf{k}) + \bar{\varphi}_<(x;\mathbf{k},\alpha)d_\alpha(\mathbf{k})$$

with $\bar{\varphi} = \varphi^\dagger\gamma^0$ [31], where $\varphi^\dagger$ is the adjoint of the spinor φ. The creation and annihilation operators are $b_n^\dagger$ and b_n for bound states, $b_\alpha^\dagger(\mathbf{k})$ and $b_\alpha(\mathbf{k})$ for positive energy scattering states and $d_\alpha^\dagger(\mathbf{k})$ and $d_\alpha(\mathbf{k})$ for negative energy scattering states.

For a neutral scalar field $\phi(x)$ in the vacuum, the construction is simpler

$$\phi(x) = \int (\varphi(x;\mathbf{k})a(\mathbf{k}) + \varphi^\dagger(x;\mathbf{k})a^\dagger(\mathbf{k}))\mathrm{d}\mu(\mathbf{k}).$$

2.3.2. The Hopf superalgebra of fields. We define V as the vector space generated by the free fields (e.g. $\psi(x)$, $\overline{\psi}(x)$ and $A^\mu(x)$ for all x in quantum electrodynamics). Then the Hopf superalgebra $\hat{H} = \mathsf{Sym}(\hat{V})$ extends to a Hopf superalgebra structure on $H = \mathsf{Sym}(V)$. The normal product of creation and annihilation operators extends to a normal product of fields, also denoted by $\vee$. For example

$$\begin{aligned}\phi(x)\vee\phi(y) = \int \mathrm{d}\mu(\mathbf{k})\mathrm{d}\mu(\mathbf{q})(&\varphi^\dagger(x;\mathbf{k})\varphi(y;\mathbf{q})a^\dagger(\mathbf{k})\vee a(\mathbf{q}) + \varphi^\dagger(x;\mathbf{k})\varphi^\dagger(y;\mathbf{q})a^\dagger(\mathbf{k})\vee a^\dagger(\mathbf{q})\\ &+\varphi(x;\mathbf{k})\varphi(y;\mathbf{q})a(\mathbf{k})\vee a(\mathbf{q}) + \varphi(x;\mathbf{k})\varphi^\dagger(y;\mathbf{q})a(\mathbf{k})\vee a^\dagger(\mathbf{q})).\end{aligned}$$

The coproduct is extended from the coproduct of $\hat{H}$. This extension uses the fact that the transformation from $\hat{V}$ to V is linear. For example

$$\Delta A_\mu(x) = A_\mu(x)\otimes 1 + 1\otimes A_\mu(x)$$

$$\Delta\psi(x) = \psi(x)\otimes 1 + 1\otimes\psi(x)$$

$$\Delta(\overline{\psi}(x)\vee\psi(y)) = \overline{\psi}(x)\vee\psi(y)\otimes 1 + 1\otimes\overline{\psi}(x)\vee\psi(y) + \overline{\psi}(x)\otimes\psi(y) - \psi(y)\otimes\overline{\psi}(x).$$

The counit of H is extended from the counit of $\hat{H}$. Thus, $\varepsilon(\phi(x)) = \varepsilon(\psi(x)) = \varepsilon(\overline{\psi}(x)) = 0$. The antipode is defined by $\gamma(u) = (-1)^n u$ if u is the normal product of n elements of V, and extended to H by linearity. H is a graded commutative and graded cocommutative Hopf superalgebra.

The $*$-structure of quantum field operators is deduced from the $*$-structure on creation and annihilation operators $a(s)^* = a^\dagger(s)$. It gives $\phi(x)^* = \phi(x)$ for a neutral scalar field and $\psi(x)^* = \psi^\dagger(x) = \overline{\psi}(x)\gamma^0$ and $\overline{\psi}(x)^* = \gamma^0\psi(x)$ for Dirac fields. The $*$-structure is related to the charge conjugation operator $\mathcal{C}$ of Dirac fields by $\mathcal{C}\psi(x)\mathcal{C}^\dagger = \mathrm{i}\gamma^2\psi(x)^*$ (see [31]).

2.4. Operator and time-ordered product of field operators

In this section we show that, by properly choosing the Laplace pairing, we can twist the normal product into the operator product or the time-ordered product. The twisted product on $\hat{H}$ or H is given by equation (4). From the coproduct of these Hopf superalgebras and the definition (4) of the twisted product, we obtain the following simple examples, valid for a, b and c in $\hat{V}$ (or V).

$$\begin{aligned}a\circ b &= a\vee b + (a|b)\\ (a\vee b)\circ c &= a\vee b\vee c + (-1)^{|b||c|}(a|c)b + (-1)^{|a||b|+|a||c|}(b|c)a\\ a\circ(b\vee c) &= a\vee b\vee c + (-1)^{|b||c|}(a|c)b + (a|b)c\\ a\circ b\circ c &= a\vee b\vee c + (a|b)c + (-1)^{|b||c|}(a|c)b + (-1)^{|a||b|+|a||c|}(b|c)a.\end{aligned} \tag{6}$$

Now we are going to specify the Laplace pairings relevant to the algebra of normal products $\hat{H}$ and the algebra of fields H.

2.4.1. The algebra of operator products $\hat{A}_O$. We call *Wightman pairing* the Laplace pairing defined as follows. For a scalar field in the vacuum the Wightman pairing is

$$(a(\mathbf{k})|a^\dagger(\mathbf{q}))_+ = \frac{\delta(\mathbf{k}-\mathbf{q})}{\rho(\mathbf{k})} \tag{7}$$

where $\rho(\mathbf{k}) = (2\pi)^{-3}m/\sqrt{\mathbf{k}\cdot\mathbf{k}+m^2}$, all the other pairings being zero. For Dirac fields we have

$$\big(b_n\big|b_p^\dagger\big)_+ = \delta_{np}$$
$$\big(b_\alpha(\mathbf{k})\big|b_\beta^\dagger(\mathbf{q})\big)_+ = \big(d_\alpha(\mathbf{k})\big|d_\beta^\dagger(\mathbf{q})\big)_+ = \delta_{\alpha\beta}\frac{\delta(\mathbf{k}-\mathbf{q})}{\rho(\mathbf{k})}$$

all the other pairings being zero. This Wightman pairing twists the normal product into the operator product and the algebra $\hat{A}_N$ becomes the algebra $\hat{A}_O$. But we need first to prove the following proposition:

Proposition 2.3. *The twisted product defined by the Wightman pairing is equal to the operator product: for any elements u and v of $\hat{A}_N$, $u \circ v = uv$.*

For the case of a scalar field, we are going to show that the operator product of two elements u and v of $\hat{A}_N$ is equal to the twisted product of these elements with the Wightman pairing (7). The proof for Dirac fields is analogous.

For two operators a and b, where $a = a(s)$ or $a = a^\dagger(s)$ and $b = a(s)$ or $b = a^\dagger(s')$, equation (6) tells us that $a \circ b = a{\vee}b + (a|b)_+$. On the other hand we know from Wick's theorem [31] that the operator product satisfies $ab = a{\vee}b + \langle 0|ab|0\rangle$ (recall that $a{\vee}b = {:}ab{:}$). The Wightman pairing was defined precisely so that $(a|b)_+ = \langle 0|ab|0\rangle$, thus $a \circ b = ab$. This equality is valid for any elements a and b in $\hat{V}$, the vector space generated by $a(s)$ and $a^\dagger(s)$.

We must now prove that $uv = u \circ v$ for any u and v in $\hat{A}_N$. This is done by using Wick's theorem [3]. Wick's theorem is very well known, so we recall it only briefly. It states that the operator product of some elements of $\hat{V}$ is equal to the sum over all possible pairs of contractions (see e.g. [32], p 209; [33], p 261; [34] p 85). A contraction[7] $\overset{\sqcap}{ab}$ is the difference between the operator product and the normal product. $\overset{\sqcap}{ab} = ab - a{\vee}b$, so that $\overset{\sqcap}{ab} = \langle 0|ab|0\rangle = (a|b)_+$.

If $u = a_1{\vee}\cdots{\vee}a_n$, Wick's theorem for bosons is proved recursively from the following identity [3]:

$$ub = u{\vee}b + \sum_{j=1}^{n}(a_j|b)_+ a_1{\vee}\cdots{\vee}a_{j-1}{\vee}a_{j+1}{\vee}\cdots{\vee}a_n. \tag{8}$$

Thus, to show that $u \circ b = ub$ we must recover equation (8) from our definition. In other words, we must prove

$$u \circ b = u{\vee}b + \sum_{j=1}^{n}(a_j|b)_+ a_1{\vee}\cdots{\vee}a_{j-1}{\vee}a_{j+1}{\vee}\cdots{\vee}a_n. \tag{9}$$

To show this, we make a recursive proof with respect to the degree of u. We recall that an element has degree k if it can be written as the normal product of k creation or annihilation operators (see section A.1). We use the definition (4) of the twisted product and the fact that $\Delta b = b \otimes 1 + 1 \otimes b$ to find

$$u \circ b = u{\vee}b + \sum (u_{(1)}|b)_+ u_{(2)}. \tag{10}$$

[7] Contractions were first used by Houriet and Kind [35].

The Wightman pairing $(u_{(1)}|b)_+$ is zero if $u_{(1)}$ is not of degree 1. According to equation (36) for Δu, this happens only for the $(1, n-1)$ shuffles. By definition, a $(1, n-1)$ shuffle is a permutation σ of $(1, \ldots, n)$ such that $\sigma(2) < \cdots < \sigma(n)$, and the corresponding terms in the coproduct of Δu are

$$\sum_{j=1}^{n} a_j \otimes a_1 \vee \cdots \vee a_{j-1} \vee a_{j+1} \vee \cdots \vee a_n.$$

Thus, we recover (9) from the Laplace identity (10) and the twisted product of u and b is the operator product of u and b. By linearity of the twisted and operator products, this can be extended to any element of $\hat{A}_N$ and we have $u \circ b = ub$ for any u in $\hat{A}_N$ and any $b \in \hat{V}$. A similar argument leads to

$$a \circ u = au \tag{11}$$

for any a in $\hat{V}$ and any u in $\hat{A}_N$. Now we proceed by induction. Assume that $u \circ v = uv$ for any $v \in \hat{A}_N$ and for u of degree $k \leqslant n$. We take now an element u of degree n and we calculate $(a \vee u) \circ v$ where a is in $\hat{V}$. We use $a \vee u = a \circ u - \sum (a|u_{(1)})_+ u_{(2)}$ and we write

$$\begin{aligned}(a \vee u) \circ v &= (a \circ u) \circ v - \sum (a|u_{(1)})_+ u_{(2)} \circ v \\ &= a \circ (u \circ v) - \sum (a|u_{(1)})_+ u_{(2)} \circ v \\ &= a \circ (uv) - \sum (a|u_{(1)})_+ u_{(2)} v = auv - \sum (a|u_{(1)})_+ u_{(2)} v\end{aligned}$$

by associativity of the twisted product and because of the recursion hypothesis and equation (11). By associativity of the operator product this can be rewritten

$$\begin{aligned}(a \vee u) \circ v &= \Big(au - \sum (a|u_{(1)})_+ u_{(2)}\Big) v = \Big(a \circ u - \sum (a|u_{(1)})_+ u_{(2)}\Big) v \\ &= (a \vee u) v.\end{aligned}$$

Therefore $u \circ v = uv$ if u is of degree $n+1$. By induction, this proves that $u \circ v = uv$ for an element u of any degree. By linearity, this shows that $u \circ v = uv$ for any u and v in $\hat{A}_N$ and the property is proved for bosons.

Adding the proper signs, the same proof shows that $u \circ v = uv$ if $\hat{A}_N$ contains boson and fermion fields.

2.4.2. Operator twisting of the algebra of fields. The Wightman pairing $(|)_+$ on the algebra $\hat{A}_N$ of normal products extends to a Laplace pairing $(|)_+$ on the algebra of fields A_N, that we also call the Wightman pairing. For scalar fields we obtain

$$(\phi(x)|\phi(y))_+ = \int \varphi(x; \mathbf{k})^\dagger \varphi(y; \mathbf{k}) \mathrm{d}\mu(\mathbf{k})$$

which can again be defined as $(\phi(x)|\phi(y))_+ = \langle 0|\phi(x)\phi(y)|0\rangle$. The Wightman pairing for the product of Dirac fields is

$$(\psi(x)|\overline{\psi}(y))_+ = \sum_n \varphi_>(x; n)\bar{\varphi}_>(y; n) + \sum_{\alpha=1}^{2} \int \varphi_>(x; \mathbf{k}, \alpha)\bar{\varphi}_>(y; \mathbf{k}, \alpha) \mathrm{d}\mu(\mathbf{k})$$

$$(\overline{\psi}(x)|\psi(y))_+ = \sum_{\alpha=1}^{2} \int \varphi_<(x; \mathbf{k}, \alpha)\bar{\varphi}_<(y; \mathbf{k}, \alpha) \mathrm{d}\mu(\mathbf{k})$$

$$(\psi(x)|\psi(y))_+ = 0 \qquad (\overline{\psi}(x)|\overline{\psi}(y))_+ = 0.$$

For scalar fields in the vacuum this gives us

$$(\phi(x)|\phi(y))_+ = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3 2\omega_k}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$
$$= \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\delta(p^2 - m^2)\theta(p^0)\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}.$$

For Dirac fields in the vacuum

$$(\psi(x)|\overline{\psi}(y))_+ = \int \frac{\mathrm{d}p}{(2\pi)^3}\delta(p^2 - m)\theta(p^0)(\gamma\cdot p + m)\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$
$$(\overline{\psi}(x)|\psi(y))_+ = -\int \frac{\mathrm{d}p}{(2\pi)^3}\delta(p^2 - m)\theta(-p^0)(\gamma\cdot p + m)\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}.$$

The proof of proposition 2.3 can be repeated to show that

Proposition 2.4. *The twisted product defined by the Wightman pairing is equal to the operator product of fields: for any elements u and v of A_N, $u \circ v = uv$.*

Therefore, the Wightman pairing twists the algebra A_N of normal products of fields into the algebra A_O of operator products of fields. Note that the Dirac operator $D = \mathrm{i}\gamma\cdot\partial_x - m$ annihilates the Wightman pairing: $D(\psi(x)|\overline{\psi}(y))_+ = D(\overline{\psi}(x)|\psi(y))_+ = 0$.

2.4.3. Time-ordered twisting of the algebra of fields. We call *Feynman pairing* the Laplace pairing defined by

$$(\phi(x)|\phi(y))_F = \theta(x^0 - y^0)(\phi(x)|\phi(y))_+ + \theta(y^0 - x^0)(\phi(y)|\phi(x))_+$$

for scalar fields and

$$(\psi_\xi(x)|\overline{\psi}_{\xi'}(y))_F = -(\overline{\psi}_{\xi'}(y)|\psi_\xi(x))_F$$
$$= \theta(x^0 - y^0)(\psi_\xi(x)|\overline{\psi}_{\xi'}(y))_+ - \theta(y^0 - x^0)(\overline{\psi}_{\xi'}(y)|\psi_\xi(x))_+$$
$$(\psi(x)|\psi(y))_F = 0 \qquad (\overline{\psi}(x)|\overline{\psi}(y))_F = 0$$

for Dirac fields. The Feynman pairing is proportional to the Feynman propagator: $(\psi(x)|\overline{\psi}(y))_F = \mathrm{i}S_F(x - y)$. In the vacuum

$$(\psi(x)|\overline{\psi}(y))_F = \mathrm{i}\int \frac{\mathrm{d}k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k\cdot(x-y)}}{\gamma\cdot k - m + \mathrm{i}\epsilon}.$$

The action of the Dirac operator on the Feynman pairing is $D(\psi(x)|\overline{\psi}(y))_F = \mathrm{i}\delta(x - y)$.

The time-ordered product satisfies the same Wick theorem as the operator product [36]. Thus the same proof can be used to show

Proposition 2.5. *The twisted product defined by the Feynman pairing is equal to the time-ordered product: for any elements u and v of $\hat{A}_N$, $u \circ v = T(uv)$.*

Therefore, the Feynman pairing twists the algebra A_N of normal products of fields into the algebra A_T of time-ordered products of fields. We saw that the twisted product is associative. Thus, the time-ordered product of free fields is associative. As far as we know, this property of time-ordered products was never pointed out explicitly.

2.4.4. $*$-structure and real Laplace pairings. The $*$-structure satisfies $(u\vee v)^* = v^*\vee u^*$. A Laplace pairing is called *real* (see [24], p 55) when the corresponding twisted product satisfies $(u \circ v)^* = v^* \circ u^*$. In this section, we investigate the properties of a real Laplace pairing. First, it can be shown that a Laplace pairing is real if and only if $(u|v)^* = (v^*|u^*)$.

In the case of $\hat{A}_O$, it can be checked that the Wightman pairing $(|)_+$ is real, because the density $\rho(\mathbf{k})$ is real. For a neutral scalar field, $\phi(x)^* = \phi(x)$ and $(\phi(x)|\phi(y))_+^* = (\phi(y)|\phi(x))_+$. Thus, the Wightman pairing is real. Similarly, for a Dirac field $(\psi(x)|\overline{\psi}(y))_+^* = (\overline{\psi}(y)^*|\psi(x)^*)_+$. Thus, these Wightman pairings are real and we have $(uv)^* = v^*u^*$, which is the expected behaviour of operator products.

The Feynman pairing $(|)_F$ corresponding to the time-ordered product is not real. However, the $*$-operation is still important because it is related to the time-reversal symmetry (see section 3.3.4).

2.4.5. Closed formulae for Wick expansion and expectation values. We present here the application of the Hopf algebra approach to the calculation of iterated products and their vacuum expectation values. Similar results were obtained for the bosonic case in [37]. To state these results we first need to define the powers Δ^k of the coproduct as $\Delta^0 a = a$, $\Delta^1 a = \Delta a$ and $\Delta^{k+1} a = (\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\Delta)\Delta^k a$. Their action is denoted by $\Delta^k a = \sum a_{(1)}\otimes\cdots\otimes a_{(k+1)}$.

For the vacuum expectation values we have now

Proposition 2.6. *For $u^1, \ldots, u^n$ in $\hat{H}$ or H we have*

$$\varepsilon(u^1 \circ\cdots\circ u^n) = \sum_{(u)} (-1)^{F_n} \prod_{j=2}^{n}\prod_{l=1}^{j-1} \big(u^l_{(j-1)}|u^j_{(l)}\big) \tag{12}$$

where the index (u) means that we sum over the required powers of the coproducts of $u^1, \ldots, u^n$ and where

$$F_n = \sum_{i=3}^{n}\sum_{j=1}^{i-1}\sum_{k=2}^{i-1}\sum_{l=1}^{k-1} |u^k_{(j)}\|u^i_{(l)}|.$$

In the case of the time-ordered product of quantum fields, the right-hand side of equation (12) is written as a sum of Feynman diagrams. Our formula is also valid for the vacuum expectation value of the operator product of fields. An example of the application of this formula to scalar fields was given in [37].

For the Wick expansion of operator products or time-ordered products of fields, we have the

Proposition 2.7. *For $u^1, \ldots, u^n$ in $\hat{H}$ or H we have*

$$u^1 \circ\cdots\circ u^n = \sum_{(u)} (-1)^{F(u)} \varepsilon\big(u^1_{(1)} \circ\cdots\circ u^n_{(1)}\big) u^1_{(2)}\vee\cdots\vee u^n_{(2)} \tag{13}$$

where

$$F(u) = \sum_{k=2}^{n}\sum_{l=1}^{k-1} |u^k_{(1)}\|u^l_{(2)}|. \tag{14}$$

In perturbative quantum field theory, this equation is used for the calculation of the S-matrix: the product $\circ$ is then the time-ordered product and $u^1 = \cdots = u^n = \mathcal{L}$, where $\mathcal{L}$ is the interaction Lagrangian of the theory. As an example, we consider the Lagrangian for the

ϕ^n theory: $\mathcal{L}(x) = \phi^n(x)$, where $\phi^n(x)$ denotes the normal product of n fields $\phi(x)$. The binomial formula gives us the coproduct of $\mathcal{L}(x)$:

$$\Delta\phi^n(x) = \sum_{k=0}^{n} \binom{n}{k} \phi^k(x) \otimes \phi^{n-k}(x)$$

and equation (13) becomes, in the usual notation

$$T(\phi^{n_1}(x_1)\cdots\phi^{n_m}(x_m)) = \sum_{i_1=0}^{n_1}\cdots\sum_{i_m=0}^{n_m}\binom{n_1}{i_1}\cdots\binom{n_m}{i_m}$$
$$\langle 0|T(\phi^{i_1}(x_1)\cdots\phi^{i_m}(x_m))|0\rangle{:}\phi^{n_1-i_1}(x_1)\cdots\phi^{n_m-i_m}(x_m){:}$$

where T is the time-ordering operator and $:u:$ stands for the normal product. This equation can be found, for example in [38, 39]. Our equation (13) is more compact and much more general: it is valid for bosons and fermions, for products of any elements u of H (and not only of $\phi^n(x)$), for operator products as well as time-ordered products.

The proof of these formulae was given in [37] for bosonic fields, so we leave to the reader the determination of the additional signs. However, we give the main lemmas that lead to them.

Lemma 2.8. *For $u^1, \ldots, u^n$ and $v^1, \ldots, v^m$ in $\hat{H}$ or H we have*

$$(u^1\vee\cdots\vee u^n|v^1\vee\cdots\vee v^m) = \sum_{(u)(v)}(-1)^{F_{nm}}\prod_{i=1}^{n}\prod_{j=1}^{m}\big(u^i_{(j)}|v^j_{(i)}\big) \tag{15}$$

where the sign $(-1)^{F_{nm}}$ is given by

$$F_{nm} = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{(u)}\left(\big|u^i_{(j)}\big|^2 + \sum_{k=1}^{i}\sum_{l=1}^{j}\big|u^i_{(j)}\big|\big|u^k_{(l)}\big|\right).$$

To obtain this equation, we used the fact that the Laplace pairing is even, so that $\big|u^i_{(j)}\big| = \big|v^j_{(i)}\big|$. Two special cases are important [26]: (i) when all u^i and v^j are in $\hat{V}$ or V and are fermionic

$$(u^1\vee\cdots\vee u^n|v^1\vee\cdots\vee v^m) = \delta_{m,n}(-1)^{n(n-1)/2}\det(u^i|v^j)$$

and (ii) when all u^i and v^j are in $\hat{V}$ or V and bosonic

$$(u^1\vee\cdots\vee u^n|v^1\vee\cdots\vee v^m) = \delta_{m,n}\,\mathrm{perm}(u^i|v^j)$$

where $\mathrm{perm}(u^i|v^j)$ is the permanent of the matrix $(u^i|v^j)$.

To calculate iterated products recursively we need the following identity:

Lemma 2.9. *For $u_1, \ldots, u_n$ in $\hat{H}$ or H,*

$$\Delta(u^1\circ\cdots\circ u^n) = \sum_{(u)}(-1)^{F(u)}u^1_{(1)}\circ\cdots\circ u^n_{(1)}\otimes u^1_{(2)}\vee\cdots\vee u^n_{(2)} \tag{16}$$

where $F(u)$ is given by equation (14).

These results illustrate the power of Hopf algebra methods to derive explicit expressions in quantum field theory.

3. Cohomology

In this section, we uncover some of the deeper mathematical structures that lie behind the twist construction of the time-ordered product and the operator product. Principally, these are Sweedler's Hopf algebra cohomology and the Drinfeld twist. These new insights in turn give us new tools for quantum field theory that will be exploited subsequently to describe interactions (section 4) and non-trivial vacua (section 5).

3.1. Cohomology of Hopf superalgebras

In this section, we review the basics of Sweedler's cohomology theory of cocommutative Hopf algebras [18] generalized by Majid [24]. We adapt it here to the Hopf superalgebra case.

3.1.1. Convolution product. Let H be a Hopf superalgebra. Consider the set $L^n(H)$ of even linear maps $H \otimes \cdots \otimes H \to \mathbb{C}$ on the n-fold tensor product of H. A linear map χ is even if $\chi(a_1, \ldots, a_n) = 0$ when $|a_1| + \cdots + |a_n|$ is odd. Let ϕ and ψ be two even maps. We define their *convolution product* as the element in $L^n(H)$ given by

$$(\phi \star \psi)(a_1, \ldots, a_n) = \sum (-1)^{\sum_{k=2}^{n} \sum_{l=1}^{k-1} |a_{k_{(1)}}||a_{l_{(2)}}|} \phi\big(a_{1_{(1)}}, \ldots, a_{n_{(1)}}\big) \psi\big(a_{1_{(2)}}, \ldots, a_{n_{(2)}}\big). \tag{17}$$

For example, the product of $\phi, \psi \in L^1(H)$ reads simply

$$(\phi \star \psi)(a) = \sum \phi(a_{(1)}) \psi(a_{(2)}).$$

For $\phi, \psi \in L^2(H)$ the product is

$$(\phi \star \psi)(a, b) = \sum (-1)^{|b_{(1)}||a_{(2)}|} \phi(a_{(1)}, b_{(1)}) \psi(a_{(2)}, b_{(2)}).$$

The convolution product makes $L^n(H)$ into an algebra. It is unital with the unit given by $e(a_1, \ldots, a_n) = \varepsilon(a_1) \cdots \varepsilon(a_n)$. Thus, a *convolution inverse* for an element $\chi \in L^n(H)$ is an element $\chi^{-1} \in L^n(H)$ such that $\chi \star \chi^{-1} = \chi^{-1} \star \chi = e$.

3.1.2. Cochains and coboundary. An *n-cochain* is an element $\chi \in L^n(H)$ such that χ is convolution invertible and counital. *Counitality* is the property

$$\chi(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n) = \varepsilon(a_1) \cdots \varepsilon(a_{i-1}) \varepsilon(a_{i+1}) \cdots \varepsilon(a_n) \tag{18}$$

for all $i \in \{1, \ldots, n\}$. We denote by $C^n(H)$ the set of n-cochains on H. The set $C^n(H)$ forms a group with the convolution product. The unit element is the cochain $e(a_1, \ldots, a_n) = \varepsilon(a_1) \cdots \varepsilon(a_n)$.

For $i = 0, \ldots, n+1$ consider the maps $\partial_i^n : C^n(H) \to C^{n+1}(H)$ defined by

$$\begin{aligned}
\big(\partial_i^n \chi\big)(a_1, \ldots, a_{n+1}) &= \chi(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \qquad \forall i \in \{1, \ldots, n\} \\
\big(\partial_0^n \chi\big)(a_1, \ldots, a_{n+1}) &= \varepsilon(a_1) \chi(a_2, \ldots, a_{n+1}) \\
\big(\partial_{n+1}^n \chi\big)(a_1, \ldots, a_{n+1}) &= \chi(a_1, \ldots, a_n) \varepsilon(a_{n+1}).
\end{aligned}$$

The map $\partial^n : C^n(H) \to C^{n+1}(H)$ defined by

$$\partial^n \chi = \big(\partial_0^n \chi\big) \star \big(\partial_2^n \chi\big) \star \cdots \star \big(\partial_1^n \chi^{-1}\big) \star \big(\partial_3^n \chi^{-1}\big) \star \cdots \tag{19}$$

is called the *coboundary* map.[8] For example

$$\partial^1\chi(a,b) = \sum \chi(a_{(1)})\chi(b_{(1)})\chi^{-1}(a_{(2)}b_{(2)}) \tag{20}$$

$$\begin{aligned}\partial^2\chi(a,b,c) = &\sum(-1)^{|a_{(1)}||a_{(2)}|+|c_{(2)}||b_{(3)}|+|b_{(1)}||b_{(2)}|}(-1)^{|b_{(1)}||b_{(3)}|+|b_{(1)}||b_{(4)}|+|b_{(2)}||b_{(4)}|+|b_{(3)}||b_{(4)}|}\\ &\times \chi(b_{(1)},c_{(1)})\chi(a_{(1)},b_{(2)}c_{(2)})\chi^{-1}(a_{(2)}b_{(3)},c_{(3)})\chi^{-1}(a_{(3)},b_{(4)}).\end{aligned} \tag{21}$$

We usually write just ∂ instead of ∂^n.

3.1.3. Cohomology groups. An n-cochain χ with the property $\partial\chi = e$ is called an *n-cocycle*. The cocycles from a subset $Z^n(H)$ of $C^n(H)$. Explicitly, the cocycle condition for a 1-cochain comes out as

$$\chi(a)\chi(b) = \chi(ab) \tag{22}$$

while the 2-cocycle condition can be written as

$$\sum(-1)^{|b_{(2)}||c_{(1)}|}\chi(b_{(1)},c_{(1)})\chi(a,b_{(2)}c_{(2)}) = \sum(-1)^{|a_{(2)}||b_{(1)}|}\chi(a_{(1)},b_{(1)})\chi(a_{(2)}b_{(2)},c). \tag{23}$$

An n-cochain χ that arises from an $(n-1)$-cochain ξ as $\chi = \partial\xi$ is called an *n-coboundary*. The coboundaries also form a subset $B^n(H)$ of $C^n(H)$.

Now assume that H is graded cocommutative. The convolution product is then commutative and $C^n(H)$ is an Abelian group. Furthermore ∂ becomes a group homomorphism and both $Z^n(H)$ and $B^n(H)$ become groups. Moreover, we then have

$$\partial\partial\xi = e$$

so that an n-coboundary is in particular an n-cocycle. Thus $B^n(H)$ is a subgroup of $Z^n(H)$ and we can form the quotient group $H^n(H) = Z^n(H)/B^n(H)$. This is called the nth cohomology group of H.[9]

3.2. Drinfeld twist

In this section, we review basic properties of Drinfeld twists due to Drinfeld [2] and Sweedler [18]. We present a version adapted to Hopf superalgebras.

We first recall the notions of comodule (representation) and comodule superalgebra. A (left) *comodule* of a Hopf algebra H is a vector space A together with a linear map $\beta : A \to H \otimes A$ such that $(\mathrm{id}\otimes\beta)\circ\beta = (\Delta\otimes\mathrm{id})\circ\beta$ and $(\varepsilon\otimes\mathrm{id})\circ\beta = \mathrm{id}$. β is called a *coaction*. For coactions we also use a modified version of Sweedler's notation with the component in the comodule underlined, $\beta(a) = \sum a_{(1)}\otimes a_{(\underline{2})}$.

Consider a comodule A of H that is at the same time a superalgebra. It is called a *comodule superalgebra* of H if product (denoted by $\cdot$) and comodule structure satisfy the following compatibility condition:

$$\sum(a\cdot b)_{(1)}\otimes(a\cdot b)_{(\underline{2})} = \sum(-1)^{|b_{(1)}||a_{(\underline{2})}|}a_{(1)}b_{(1)}\otimes a_{(\underline{2})}\cdot b_{(\underline{2})}.$$

[8] Note that the group operation appearing in the definition of the coboundary map is not the vector space addition as, e.g., in Hochschild cohomology. Nevertheless, in order to give rise to a cohomology the group operation has to be Abelian, see below.

[9] Of course only in this case of graded cocommutativity is the word 'cohomology' fully justified. Indeed this is the only situation of interest in the present paper. However, important elements of the cohomology remain applicable in the case of non (graded) cocommutative Hopf algebras.

3.2.1. Twisting Hopf superalgebras and comodules. Let H be a Hopf superalgebra and $\chi \in Z^2(H)$ a 2-cocycle on H. There is a new Hopf superalgebra H_χ, the *twist* of H by χ. H_χ has the same unit, counit and coproduct as H but a different product and antipode. The new product is given by

$$a \bullet b = \sum (-1)^{|b_{(1)}|(|a_{(2)}|+|a_{(3)}|)+|b_{(2)}||a_{(3)}|} \chi(a_{(1)}, b_{(1)}) a_{(2)} b_{(2)} \chi^{-1}(a_{(3)}, b_{(3)}). \quad (24)$$

It turns out that a twist can be applied not only to the Hopf superalgebra itself but also to its comodules[10]. If a comodule A is a comodule superalgebra the twist affects its superalgebra structure. A is twisted into a comodule superalgebra A_χ of H_χ with the new associative product $\circ$ defined by

$$a \circ b = \sum (-1)^{|b_{(1)}||a_{\underline{(2)}}|} \chi(a_{(1)}, b_{(1)}) a_{\underline{(2)}} \cdot b_{\underline{(2)}}. \quad (25)$$

If H is graded cocommutative, its product remains unmodified under a twist and H_χ is the same as H. However, this is not true for a comodule superalgebra A of H. In general A_χ is different from A even in the graded cocommutative case. Indeed, the difference between twisted comodule superalgebras is related to the cohomology of H as follows:

Proposition 3.1. *Let H be a graded cocommutative Hopf superalgebra, A a left H-comodule superalgebra and $\eta, \chi \in Z^2(H)$. If η and χ are cohomologous in the sense of $\eta = \partial\rho \star \chi$ for $\rho \in C^1(H)$, then A_η and A_χ are isomorphic as comodule superalgebras. An isomorphism $T : A_\eta \to A_\chi$ is explicitly given by $T(a) = \sum \rho(a_{(1)}) a_{\underline{(2)}}$.*

If $A = H$ with the coaction given by the coproduct, the converse is also true. That is, if A_η and A_χ are isomorphic as comodule superalgebras then η and χ are cohomologous.

A proof based on [24] can be found in appendix B.2.

3.2.2. Twisting and $*$-structure. Suppose that H is a Hopf $*$-superalgebra in the sense of appendix A.2 and A a graded left comodule of H equipped with an involution $* : A \to A$. Then we call A a $*$-comodule if the coaction satisfies

$$\sum (a^*)_{(1)} \otimes (a^*)_{\underline{(2)}} = \sum (-1)^{|a_{(1)}||a_{\underline{(2)}}|} (a_{(1)})^* \otimes (a_{\underline{(2)}})^*. \quad (26)$$

In the same way, we can define a $*$-comodule superalgebra A. In this case, we can furthermore investigate under which circumstances a 2-cocycle χ gives rise to a twisted comodule superalgebra A_χ which is again a $*$-superalgebra. Indeed it is straightforward to verify that a sufficient condition on χ for this to happen is

$$\chi(a^*, b^*) = \overline{\chi(b, a)}. \quad (27)$$

A 2-cocycle satisfying this property we call *real*. Our definition is inspired by the analogous definition for coquasitriangular structures in the literature [24, definition 2.2.8]. It extends the definition for real Laplace pairing given in section 2.4.4.

3.3. Cohomology in quantum field theory

We are now ready to interpret and extend the results of section 2 from a cohomological point of view.

[10] The twist gives rise to an equivalence of the monoidal categories of comodules of H and H_χ [2]. This is explained in detail in [10].

3.3.1. The twisted field algebras as Drinfeld twists. Recall from section 2.3 that the field operators with the normal product form a Hopf superalgebra H. Forgetting the coproduct and counit we also denoted this superalgebra by A_N. Then we discovered in section 2.4 that for certain Laplace pairings on this Hopf algebra we obtain new twisted algebras A_O, A_T using (4) which reproduce either the operator or the time-ordered product.

Armed with the tools of Hopf algebra cohomology (section 3.1) and the Drinfeld twist (section 3.2) we see more clearly what is going on. Namely, the Laplace pairings give rise to Drinfeld twists of A_N as a comodule superalgebra of H. Let us clearly explain this step by step.

Firstly, a Laplace pairing (in the graded cocommutative case) is in particular a 2-cocycle.

Lemma 3.2. *Let H be a graded cocommutative Hopf superalgebra. Then a Laplace pairing, i.e. a map $\chi : H \otimes H \to \mathbb{C}$ with the properties (1), (2) and (3) is a 2-cocycle.*

Proof. Equation (3) is the counitality property (18). Let η be the linear map $H \otimes H \to \mathbb{C}$ defined by $\eta(a, b) := \chi(\gamma(a), b)$. It is elementary to check that η is the convolution inverse of χ. Thus, χ is a 2-cochain. Finally, using graded cocommutativity the cocycle condition (23) readily follows from equations (1) and (2). □

Secondly, A_N is a comodule superalgebra of H. Indeed, any Hopf superalgebra has a comodule superalgebra which is just a copy of itself. The coaction to be taken is the coproduct, i.e. $\sum a_{(1)} \otimes a_{(2)} := \sum a_{(1)} \otimes a_{(2)}$ in the notation introduced above. A Laplace pairing χ on H, being a 2-cocycle, gives rise to a Drinfeld twist of H according to (24). Since H is graded cocommutative the twisted Hopf superalgebra H_χ is identical to the untwisted one. χ also gives rise to an induced twist of the comodule superalgebra A_N according to (25). Since the coaction is given by the coproduct we recover the initial twist formula (4) with a new interpretation. This also explains why the twisted superalgebras A_O and A_T are no longer Hopf algebras: A_N was not considered a Hopf algebra from the beginning, despite the 'accident' $A_N = H$.[11]

Recall that apart from the superalgebra structure we use one more piece of the Hopf algebra structure of H on A_N. This is the map $\varepsilon : A_N \to \mathbb{C}$ which is the counit on H. As was shown in proposition 2.2 it plays the role of the vacuum expectation value. The twisted superalgebras A_O and A_T inherit the map ε without change[12]. ε continues to play the role of the vacuum expectation value. Only equipped with ε do the superalgebras carry the full information of quantum field theory. Superalgebras such as A_N, A_O, A_T, which carry the additional structure of a linear function $A \to \mathbb{C}$ are called *augmented* superalgebras.

For the $*$-structure, we remark that A_N and H could in principle have turned out to have different $*$-structures. What is important is only that A_N is a $*$-comodule superalgebra of H in the sense of (26). The results of section 2, however, imply that putting the same $*$-structure on H and A_N (namely that of the † operation in A_N) leads to consistent results.

3.3.2. Cohomology of $\mathsf{Sym}(V)$. We turn to cohomological aspects of the relevant Hopf superalgebra H of field operators. Recall that H has the structure of the graded symmetric

[11] One might envision applications to quantum field theory where H is different from A_N, but this is beyond the scope of the present paper.

[12] This is perfectly justified from the Drinfeld twist point of view. Morally speaking, the twist does not affect comodule structures and comodule maps as such but the tensor product of comodules (and hence a comodule algebra structure as its definition involves a tensor product of comodules). The deeper meaning of this lies in the fact that the twist gives rise to a monoidal equivalence of comodule categories. This equivalence is mediated by a functor that transforms objects and morphisms trivially and only tensor products non-trivially. See section 2 of [10] for a more explicit exposition of these facts.

Hopf superalgebra $\mathsf{Sym}(V)$, where V is the space of field operators. The cohomology groups of $\mathsf{Sym}(V)$ (in the bosonic case) are discussed in appendix B.1. More relevant for the application to quantum field theory, are the following results about the structure of cochains. The proofs of the lemmas are elaborated in appendix B.2.

Let us write $V = V_0 \oplus V_1$ for V_0 the even (bosonic) and V_1 the odd (fermionic) part of V. To ease notation we simply write C^n, Z^n and B^n for cochains, cocycles and coboundaries of $\mathsf{Sym}(V)$. Denote by N^1 the set of 1-cochains of $\mathsf{Sym}(V)$ that vanish on the subspace $V \subset \mathsf{Sym}(V)$. As is seen immediately they form a subgroup of C^1.

Lemma 3.3. *C^1 is equal to the direct product $Z^1 \times N^1$ of its subgroups. This implies that $\partial^1 : N^1 \to B^2$ is invertible, i.e. is an isomorphism of groups.*

We call a 2-cochain χ *symmetric* if it satisfies the property

$$\chi(b, a) = (-1)^{|a||b|}\chi(a, b) \tag{28}$$

for all $a, b \in \mathsf{Sym}(V)$. One easily checks that the symmetric 2-cochains form a subgroup C^2_{sym} of C^2.

Lemma 3.4. *$B^2 = Z^2_{\mathrm{sym}}$. That is, the 2-coboundaries are precisely the symmetric 2-cocycles.*

By lemma 3.2, a Laplace pairing as defined in section 2.1.2 is in particular a 2-cocycle. Furthermore, the convolution product of Laplace pairings is again a Laplace pairing. Thus, they form a subgroup of Z^2 which we will call R^2.

The introduction of something like 'antisymmetric' 2-cochains is less straightforward. We limit ourselves here to Laplace pairings. We call a Laplace pairing *antisymmetric* if it satisfies the property

$$\chi(w, v) = -(-1)^{|v||w|}\chi(v, w) \tag{29}$$

for all $v, w \in V$. The antisymmetric Laplace pairings form a subgroup R^2_{asym} of R^2.

Lemma 3.5. *Z^2 is equal to the direct product $B^2 \times R^2_{\mathrm{asym}}$ of its subgroups.*

From lemmas 3.3 and 3.5 it follows that the first two cohomology groups are given by $H^1 = N^1$ and $H^2 = R^2_{\mathrm{asym}}$, but we shall not need these results here.

3.3.3. The operator product. We saw in section 2.4 that the operator product of quantum field theory, i.e. the product on A_O, is induced by a twist of A_N with the Wightman pairing $(|)_+$ on H. The fact that A_O is not commutative and thus not isomorphic to A_N is nicely reflected in the cohomology. Namely, the 2-cocycle $(|)_+$ is not symmetric in the sense of (28) and thus by lemma 3.4 it is not a 2-coboundary. Hence by proposition 3.1 (take $\eta = e$, the unit cochain, and $\chi = (|)_+$) the algebras A_N and A_O cannot be isomorphic.

Since the Wightman pairing $(|)_+$ is real in the sense of (27) the superalgebra A_O is a $*$-superalgebra as is A_N. This was already remarked in section 2.4.4.

3.3.4. The time-ordered product. According to section 2.4 the time-ordered product is obtained from the Feynman pairing $(|)_F$, twisting the superalgebra A_N into the superalgebra A_T. As is well known these superalgebras A_N and A_T are isomorphic as superalgebras. From the cohomological point of view this emerges as follows. The Feynman pairing $(|)_F$ is symmetric in the sense of (28) and thus by lemma 3.4 it is a 2-coboundary. Hence by proposition 3.1 the algebras A_N and A_T are isomorphic.

However, this does *not* mean that the 'deformation' of A_N into A_T is trivial from the point of view of quantum field theory. Recall that the information of quantum field theory is not contained in A_T alone, but crucially requires a map $\varepsilon : A_T \to \mathbb{C}$ describing the vacuum expectation value. That is, we are dealing with augmented superalgebras and as such (A_N, ε) and (A_T, ε) are not isomorphic. In particular, for an isomorphism $T : A_N \to A_T$ the composition $\varepsilon \circ T$ is different from ε.

Proposition 3.1 (take $\eta = e$ and $\chi = (|)_F$) not only tells us that A_N and A_T are isomorphic but even provides us with one explicit isomorphism $T : A_N \to A_T$ for each 1-cochain ρ that satisfies $\chi = (\partial\rho)^{-1}$. The choice of such 1-cochains ρ is parametrized by a 1-cocycle. We can select a unique 1-cochain ρ by imposing suitable conditions on the associated isomorphism T. A natural choice is to demand T to act identically on V, i.e. $T(v) = v$ for all $v \in V$. This is motivated by giving T the role of a time-ordering operation. Since T is given according to proposition 3.1 by $T(a) = \sum \rho(a_{(1)})a_{\underline{(2)}}$ this would imply $\rho(v) = 0$ for all $v \in V$, i.e. that $\rho \in N^1$. Lemma 3.3 implies that we can indeed choose ρ to lie in N^1 and furthermore, that this determines ρ uniquely. This gives rise to the following result.

Proposition 3.6. *Let χ be a 2-coboundary on H. Denote by $\circ$ the induced twisted product on the twisted comodule superalgebra A_T. There exists a unique 1-cochain ρ such that $(\partial\rho)^{-1} = \chi$ and $\rho(v) = 0$ for all $v \in V$. The superalgebra isomorphism $T : A_N \to A_T$ given by $T(a) = \sum \rho(a_{(1)})a_{\underline{(2)}}$ satisfies*

$$T(v_1 \vee \cdots \vee v_n) = v_1 \circ \cdots \circ v_n \tag{30}$$

for $v_1, \ldots, v_n \in V$.

Proof. As already mentioned $\rho(v) = 0$ for $v \in V$ implies $T(v) = v$. Thus T being an isomorphism implies $T(v_1 \vee \cdots \vee v_n) = T(v_1) \circ \cdots \circ T(v_n) = v_1 \circ \cdots \circ v_n$. This is all that remained to be shown. □

Note that we have formulated the proposition in a slightly more general way than required, by replacing the Feynman pairing with a general 2-coboundary. As desired, equation (30) can be interpreted as a realization of the time-ordering operation of quantum field theory as a superalgebra isomorphism between A_N and A_T.

By definition of T the 1-cochain ρ has the property $\rho = \varepsilon \circ T$. This implies that a vacuum expectation value can be expressed directly in terms of ρ.

Corollary 3.7.

$$\begin{aligned}\langle 0|\phi(x_1) \circ \cdots \circ \phi(x_n)|0\rangle &= \varepsilon(\phi(x_1) \circ \cdots \circ \phi(x_n)) \\ &= \varepsilon(T(\phi(x_1) \vee \cdots \vee \phi(x_n))) \\ &= \rho(\phi(x_1) \vee \cdots \vee \phi(x_n)).\end{aligned} \tag{31}$$

In this sense, ρ encodes directly the free n-point functions.

While we were so far only concerned with the definition of ρ we turn now to its computation. Due to lemma 3.3, $\rho \in N^1(H)$ is determined completely by $(|)_F = (\partial\rho)^{-1}$ as ∂^1 is invertible on $N^1(H)$. Indeed we can use (20) for a recursive definition of ρ. Namely, set $\rho(1) = 1$, $\rho(v) = 0$ and $\rho(v \vee w) = (v|w)_F$. Then define ρ recursively on subspaces of A_N of increasing degree by

$$\rho(a \vee b) = \sum (a_{(1)}|b_{(1)})_F \rho(a_{\underline{(2)}})\rho(b_{\underline{(2)}}).$$

As already mentioned in section 2.4.4 the Feynman pairing $(|)_F$ is not real and A_T is indeed not a $*$-superalgebra. Nevertheless, the involution * can be combined with the time-ordering map T in an interesting way. Namely, consider the map $T^*(a) := (T(a^*))^*$ for $a \in A_N$. This anti-time-ordering operator was first considered by Dyson [40] and plays an important role in non-equilibrium quantum field theory [41–43]. Our definition of the anti-time-ordering operator follows [44]. It yields (here for fermionic fields ψ)

$$T^*(\psi(x)\vee\psi^\dagger(y)) = \theta(y^0 - x^0)\psi(x)\psi^\dagger(y) - \theta(x^0 - y^0)\psi^\dagger(y)\psi(x).$$

In other words, T^* orders the fields by decreasing times from right to left. Defining the 1-cochain $\rho^* = \varepsilon \circ T^*$ we see that the anti-time-ordered product is a twisted product via the Laplace pairing $(|)_{T^*} = (\partial\rho^*)^{-1}$. More explicitly (again for the example of fermionic fields) this Laplace pairing is determined by $(\psi(x)|\psi^\dagger(y))_{T^*} = \overline{(\psi(y)|\psi^\dagger(x))_F}$. The map T^* thus becomes an algebra isomorphism $T^* : A_N \to A_{T^*}$, with A_{T^*} the algebra of field operators with the anti-time-ordered product.

In non-relativistic quantum theory, time-reversal symmetry is implemented by complex conjugation [45, 46]. In relativistic quantum field theory, the time-reversal operator Θ acts on fermion fields by $\Theta(\psi(x^0, \mathbf{x})) = \mathrm{i}\gamma^1\gamma^3\psi(-x^0, \mathbf{x})$, which does not involve the $*$-structure. The time-reversal operator relates the time-orderings by $T(\Theta(a)) = \Theta(T^*(a))$ for $a \in A_N$.

4. Interactions

Up to now we have exclusively dealt with free quantum field theory. In this section, we extend our treatment to interacting fields. On the one hand we will show how our approach to quantum field theory links up with standard perturbation theory. On the other hand we will discuss implications for interacting quantum field theory in general, and possible connections to non-perturbative approaches.

4.1. Standard perturbation theory

Introducing interactions in the standard perturbative way is straightforward, given the free n-point functions. Let us generically denote field operators by $\phi(x)$, leaving out internal indices.

Following the usual perturbation theory we write the action as $S = S_0 + \lambda S_{\text{int}}$, where S_0 is the kinetic term and λ the coupling constant. Following a path integral notation the interacting n-point functions are given by

$$\begin{aligned}\langle 0|T(\phi_{\text{int}}(x_1), \ldots, \phi_{\text{int}}(x_n))|0\rangle &= \frac{\int \mathcal{D}\phi\phi(x_1)\cdots\phi(x_n)\,\mathrm{e}^{\mathrm{i}S_0+\mathrm{i}\lambda S_{\text{int}}}}{\int \mathcal{D}\phi\,\mathrm{e}^{\mathrm{i}S_0+\mathrm{i}\lambda S_{\text{int}}}}\\ &= \frac{\sum_k \frac{1}{k!}(\mathrm{i}\lambda)^k\langle 0|T\big(\phi(x_1)\vee\cdots\vee\phi(x_n)\vee S_{\text{int}}^{\vee k}\big)|0\rangle}{\sum_k \frac{1}{k!}(\mathrm{i}\lambda)^k\langle 0|T\big(S_{\text{int}}^{\vee k}\big)|0\rangle}\\ &= \frac{\sum_k \frac{1}{k!}(\mathrm{i}\lambda)^k\rho\big(\phi(x_1)\vee\cdots\vee\phi(x_n)\vee S_{\text{int}}^{\vee k}\big)}{\sum_k \frac{1}{k!}(\mathrm{i}\lambda)^k\rho\big(S_{\text{int}}^{\vee k}\big)}.\end{aligned} \tag{32}$$

Here, $S_{\text{int}}^{\vee k}$ denotes the k-fold normal product of S_{int} with itself and $S_{\text{int}}^{\vee 0} = 1$. Alternatively, in terms of an S-matrix

$$\mathcal{S} = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{i}\lambda)^k(T(S_{\text{int}}))^{\circ k}$$

we obtain

$$\langle 0|T(\phi_{\text{int}}(x_1), \ldots, \phi_{\text{int}}(x_n))|0\rangle = \frac{\varepsilon(\phi(x_1)\circ\cdots\circ\phi(x_n)\circ\mathcal{S})}{\varepsilon(\mathcal{S})} \tag{33}$$

where $\circ$ denotes the time-ordered product.

We emphasize that the difference from conventional approaches is not only notational. While the path integral method is capable of the evaluation of equation (32), it is usually evaluated by recursive means using functional derivatives. Such algorithms were demonstrated to be computationally ineffective [16, 17]. In such a recursion, terms occur which cancel out during further steps of evaluation in the recursion. Since these terms may even contain divergent integrals which need to be renormalized, this can cause confusion and wastes labour. In contrast to this finding, the Hopf algebraic version equation (33) can be evaluated directly using the formulae provided in section 2.4.5. The remarkable fact behind these formulae is that they are explicit and efficient for any order. It can be shown that they yield exactly as many terms as are potentially non-zero. Especially one may note that no cancellations take place, as long as no further symmetries are encountered in the pairings involved in the employed twists. The formulae (12) and (13) prove to be algorithmically optimal in this sense.

4.2. Beyond perturbation theory

We turn now to general considerations of quantum field theory beyond any perturbation theory. We consider the same field operators and thus the same Hopf superalgebra H and superalgebra A_N as before. Recall that all the information of a quantum field theory (interacting or not) is encoded in the n-point functions. We write generically

$$\rho_{\mathrm{int}}(\phi(x_1)\vee\cdots\vee\phi(x_n)) := \langle 0|T(\phi_{\mathrm{int}}(x_1),\ldots,\phi_{\mathrm{int}}(x_n))|0\rangle.$$

From this point of view, the set of n-point functions is nothing but a 1-cochain ρ_{int} on H (since $\langle 0|1|0\rangle = 1$).

For the free theory we saw in section 3.3.4 that the Feynman pairing $(|)_F$ leads to a 1-cochain ρ that encodes directly the n-point functions (31). ρ was the 1-cochain in $N^1(H)$ determined by the property $(|)_F = (\partial\rho)^{-1}$. Indeed, we can turn this argument around:

Proposition 4.1. *Let $\rho \in N^1(\mathsf{Sym}(V))$, then $\chi = (\partial\rho)^{-1}$ is a 2-cocycle which induces a twisted product $\circ$ in $\mathsf{Sym}(V)$ as a comodule superalgebra over itself with the property*

$$\rho(v_1\vee\cdots\vee v_n) = \varepsilon(v_1\circ\cdots\circ v_n)$$

for $v_1,\ldots,v_n \in V$.

Proof. By proposition 3.1, $T(a) = \sum \rho(a_{(1)})a_{\underline{(2)}}$ is a superalgebra isomorphism between the original and the twisted superalgebras. The statement is then obtained by applying the counit to (30) with the proof as in proposition 3.6. □

This means, given an arbitrary set of n-point functions ρ_{int} satisfying $\rho_{\mathrm{int}}(1) = \langle 0|1|0\rangle = 1$ and $\rho_{\mathrm{int}}(\phi(x)) = \langle 0|\phi_{\mathrm{int}}(x)|0\rangle = 0$, we can construct a twisted product $\circ$ which recovers these n-point functions. This product is thus the time-ordered product of interacting fields. The 2-cocycle inducing this product is simply $\chi_{\mathrm{int}} := (\partial\rho_{\mathrm{int}})^{-1}$. We can view this as a kind of (algebraic) reconstruction result in the spirit of Streater and Wightman [47], although for the time-ordered and not the operator product.

Furthermore, according to proposition 3.1 and analogous to section 3.3.4 we obtain an isomorphism of superalgebras $T_{\mathrm{int}} : A_N \to A_{T,\mathrm{int}}$ via $T_{\mathrm{int}}(a) := \sum \rho_{\mathrm{int}}(a_{(1)})a_{\underline{(2)}}$. This isomorphism might be viewed as an interacting time ordering, i.e. it takes care at the same time of the interaction and the time ordering. Thus, we may write

$$\langle 0|T(\phi_{\mathrm{int}}(x_1),\ldots,\phi_{\mathrm{int}}(x_n))|0\rangle = \varepsilon(T_{\mathrm{int}}(\phi(x_1)\vee\cdots\vee\phi(x_n))).$$

What we have thus shown is that not only a free quantum field theory can be completely encoded in a 2-cocycle on H, but *any* quantum field theory defined through polynomial n-point functions can thus be encoded (provided its 1-point functions vanish). What is more, whether the theory is free or not corresponds to a simple property of the 2-cocycle. We define *free* here to mean that n-point functions factorize into 2-point functions according to (12).

Proposition 4.2. *Let a quantum field theory be given in terms of $H = \mathsf{Sym}(V)$ and a 2-cocycle χ on H inducing the interacting time-ordered product. Then the theory is free if and only if χ is a Laplace pairing.*

The proof is straightforward now. If χ is a Laplace pairing the n-point functions are determined by the 2-point functions according to formula (12) and thus the theory is free. Conversely, if the theory is free formula (12) holds and we can thus construct a Laplace pairing that reproduces the n-point functions. Since these determine χ uniquely it must be identical to the constructed Laplace pairing.

5. Non-trivial vacua

In this section, we illustrate again the computational power of Hopf algebras by solving a problem of quantum chemistry.

A *state* is a linear map ρ from $\hat{A}_N$ to $\mathbb{C}$ such that $\rho(1) = 1$ and $\rho(u^*u) \geqslant 0$ for any element u of $\hat{A}_N$ [48] (recall that u^*u is the operator product of u^* and u). The *pure states* are states of the form $\rho(u) = \langle\psi|u|\psi\rangle$, where $|\psi\rangle$ is a vector of the Fock space $\mathcal{F}$, and the states that are not pure are called *mixed states*. They can be written as a weighted sum of pure states. Physically relevant pure states are such that $\langle\psi|u|\psi\rangle = 0$ if u contains an odd number of Dirac fields. Thus we consider states $\rho(u)$ which are zero if u contains an odd number of Dirac fields. Since ρ is linear, even and $\rho(1) = 1$, a state is 1-cochain.

In [19], Kutzelnigg and Mukherjee study the following problem of quantum chemistry. Assume that a quantum system is described by a state ρ, is it possible to define normal products adapted to ρ? The usual normal products are adapted to the vacuum because $\epsilon(u) = \langle 0|u|0\rangle$ is zero if u has no scalar part (i.e. no part proportional to 1). To adapt a normal product to a state ρ, we start from an element $u \in \hat{A}_N$ and we want to find an element $\tilde{u}$ such that $\rho(\tilde{u}) = \epsilon(u)$, so that the state ρ can be considered as the new vacuum of the system. Kutzelnigg and Mukherjee investigated this problem for pure states and they solved it for elements u which are the normal product of a small number of creation and annihilation operators. The Hopf algebra methods will enable us to solve it for general states and to give formulae that are valid for any $u \in \hat{A}_N$.

The solution of this problem is quite simple. We just have to define

$$\tilde{u} = \sum \rho^{-1}(u_{(1)})u_{(2)} \tag{34}$$

because $\rho(\tilde{u}) = \sum \rho^{-1}(u_{(1)})\rho(u_{(2)}) = (\rho^{-1} \star \rho)(u) = \varepsilon(u)$. Note that formula (34) is another instance of a T-operator, where ρ is replaced by ρ^{-1}.

Let us give a few examples. In this section, we consider Dirac fields, which are relevant for this type of application. The convolution inverse ρ^{-1} is even because ρ is even. It can be computed recursively by (see [49], p 259)

$$\rho^{-1}(1) = 1$$

$$\rho^{-1}(u) = -\rho(u) - \sum{}' \rho^{-1}(u_{(1)})\rho(u_{(2)})$$

where $\sum' u_{(1)} \otimes u_{(2)} := \Delta u - 1 \otimes u - u \otimes 1$ for $u \in \hat{A}_N$ and u contains the product of two or more creation or annihilation operators. The first examples for Dirac fields are

$$\rho^{-1}(a\vee b) = -\rho(a\vee b)$$

$$\rho^{-1}(a\vee b\vee c\vee d) = -\rho(a\vee b\vee c\vee d) + 2\rho(a\vee b)\rho(c\vee d) - 2\rho(a\vee c)\rho(b\vee d) + 2\rho(a\vee d)\rho(b\vee c)$$

where a, b, c, d are Dirac creation or annihilation operators. From ρ^{-1} and equation (34) we can calculate the first adapted normal products of Dirac fields.

$$\tilde{a} = a$$
$$\widetilde{a\vee b} = a\vee b - \rho(a\vee b)$$
$$\widetilde{a\vee b\vee c} = a\vee b\vee c - \rho(a\vee b)c + \rho(a\vee c)b - \rho(b\vee c)a$$
$$\begin{aligned}\widetilde{a\vee b\vee c\vee d} = {} & a\vee b\vee c\vee d - \rho(a\vee b)c\vee d + \rho(a\vee c)b\vee d - \rho(b\vee c)a\vee d \\ & - \rho(a\vee d)b\vee c + \rho(b\vee d)a\vee c - \rho(c\vee d)a\vee b - \rho(a\vee b\vee c\vee d) \\ & + 2\rho(a\vee b)\rho(c\vee d) - 2\rho(a\vee c)\rho(b\vee d) + 2\rho(a\vee d)\rho(b\vee c).\end{aligned}$$

It can be checked that these $\tilde{u}$ coincide with the adapted normal products defined in [19].

The second question posed by Kutzelnigg and Mukherjee is: once we have defined adapted normal products, how can we express their operator products? Again, the Hopf algebra methods yield a complete answer. Starting from $\tilde{u}$ and $\tilde{v}$ we look for a 2-cocycle χ such that

$$\tilde{u}\tilde{v} = \sum (-1)^{|u_{(2)}||v_{(1)}|} \chi(u_{(1)}, v_{(1)}) \widetilde{u_{(2)}\vee v_{(2)}}. \tag{35}$$

The answer is now expected by the reader: $\chi = (\partial\rho^{-1}) \star (.|.)_+$.

At this stage, it will be useful to give a few examples to show the kind of expressions that are obtained by a direct calculation. According to lemma 3.4, $\partial\rho^{-1}(u, v) = (-1)^{|u||v|}\partial\rho^{-1}(v, u)$. The value of $\partial\rho^{-1}(u, v)$ for the simplest elements of $\hat{A}_N$ is, again for Dirac fields,

$$\partial\rho^{-1}(a, b) = \rho(a\vee b)$$
$$\partial\rho^{-1}(a\vee b, c\vee d) = \rho(a\vee b\vee c\vee d) - \rho(a\vee b)\rho(c\vee d)$$
$$\begin{aligned}\partial\rho^{-1}(a, b\vee c\vee d) &= \rho(a\vee b\vee c\vee d) - \rho(a\vee b)\rho(c\vee d) + \rho(a\vee c)\rho(b\vee d) - \rho(b\vee c)\rho(a\vee d) \\ &= \partial\rho^{-1}(a\vee b\vee c, d).\end{aligned}$$

Now we can calculate χ as

$$\chi(u, v) = \sum \partial\rho^{-1}(u_{(1)}, v_{(2)})(u_{(2)}|v_{(1)})_+ = \sum \partial\rho^{-1}(u_{(2)}, v_{(1)})(u_{(1)}|v_{(2)})_+.$$

A few example computations might be appropriate

$$\chi(a, b) = \rho(a\vee b) + (a|b)_+$$
$$\chi(a\vee b\vee c, d) = \partial\rho^{-1}(a\vee b\vee c, d)$$
$$\begin{aligned}\chi(a\vee b, c\vee d) = {} & \partial\rho^{-1}(a\vee b, c\vee d) + (a\vee b|c\vee d)_+ - \rho(a\vee c)(b|d)_+ + \rho(a\vee d)(b|c)_+ \\ & + \rho(b\vee c)(a|d)_+ - \rho(b\vee d)(a|c)_+.\end{aligned}$$

Finally, we define the product of $\tilde{u}$ and $\tilde{v}$ by $\tilde{u}\tilde{v} = \sum \partial\rho^{-1}(u_{(2)}, v_{(1)})\widetilde{u_{(1)}v_{(2)}}$. This operator product satisfies (35) and a few simple examples of it are

$$\tilde{a}\tilde{b} = \widetilde{a\vee b} + \rho(a\vee b) + (a|b)_+$$
$$(\widetilde{a\vee b})\tilde{c} = \widetilde{a\vee b\vee c} - (a|c)_+\tilde{b} + (b|c)_+\tilde{a} + \rho(b\vee c)\tilde{a} - \rho(a\vee c)\tilde{b}.$$

As illustrated in [19], the combinatorial complexity of the products increases very quickly. The Hopf algebraic concepts provide powerful tools to manipulate these products.

Moreover, this example coming from quantum chemistry shows that the convolution of a Laplace pairing by a 2-coboundary is a very natural quantum object. Lemma 3.5 says that all 2-cocycles can be obtained by such a convolution. The adapted normal product can be understood as inducing a transformation of the augmentation. This parallels the situation discussed in section 3.3.4, as exhibited in equation (31).

The basis change defined by equation (34) appears not only in quantum chemistry, but also in the context of quantum field theory in curved spacetime. In [50] (theorem 5.1), it is shown that two Wick monomials that can define a quantum field theory in curved spacetime are related by equation (34).

6. Conclusions and outlook

We conclude this paper by sketching possible future developments and applications of the presented framework.

From a computational point of view, the Hopf algebra structure presented in this paper was already used to solve two old problems of many-body theory: the hierarchy of Green functions for systems with initial correlation [51] and the many-body generalization of the crystal-field method [52]. More results along these lines might be expected.

In an interesting parallel development Kreimer discovered that the combinatorics of Bogoliubov's recursion formula of renormalization and Zimmermann's solution can be expressed in Hopf algebraic terms [53]. This was further elaborated by Connes, Kreimer and Pinter [53–55]. In particular, it was also shown how this leads to computational improvements compared to traditional methods [56]. It should be possible to connect this Hopf algebraic structure associated with the perturbative Feynman diagram expansion with the algebraic framework introduced in this paper. Indeed, Pinter's work might be seen as pointing in this direction [55]. More concrete steps for establishing such a connection were performed in [57].

Looking back at section 3.2 there are really three ingredients to the twist, the Hopf superalgebra H, the comodule (super)algebra A and the 2-cocycle χ. In the case explored in this paper, however, A and H are really the same, namely the algebra of normal ordered field operators (section 3.3.1). However, the fact that we obtain deformations of this (super)algebra only depends on the choice of A. In this respect the fact that H may also be taken to be this superalgebra (extended to a Hopf superalgebra) is an 'accident'.

One may ask whether a different choice for H (and thus also for the cocycle) may lead to any interesting constructions for QFT. This is indeed the case as evidenced by [11]. There, in essentially the same twist deformation construction for QFT the Hopf algebra H was chosen to correspond to the group of translations of Minkowski space. A certain cocycle then yields QFT on noncommutative spacetimes of the Moyal type.

Thus, the same twist procedure unifies *a priori* rather different and unrelated constructions in QFT. It seems likely that there is rather more to discover. It should be mentioned in this context that it is straightforward to consider the Hopf algebraic analogues of group products such as direct or semidirect ones. Moreover, in general the twist does not leave H invariant. In particular, it can lead to genuine quantum group symmetries even if the initial objects are group symmetries. See [11, section 4.4] for examples.

Another interesting direction for generalizing our approach is to nonlinear QFT. To this end note that H (and A_N) may be seen as the algebra of polynomial functions on field configurations. Considering a nonlinear field theory the analogue would be an algebra of functions on field configurations generated by suitably chosen modes. At least for compact group target spaces this is rather straightforward by employing the Peter–Weyl decomposition. Of course H would no longer be cocommutative if the respective group is non-Abelian. This implies for example that part of the cohomology theory no longer works. However, the crucial part, namely the Drinfeld twist, generalizes to this case. This offers perspectives to describe quantized nonlinear QFTs through our approach.

Not elaborated in the course of this work was the intimate connection of Hopf algebraic methods to combinatorics and group representations. The invariant theoretic content of Hopf superalgebraic methods was studied in [26]. The connection of Hopf algebra cohomology to branching laws, group representations and symmetric functions was investigated in [58]. From these contacts to other fields of mathematics one expects further insight into the structure of QFT. This will be considered elsewhere.

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Appendix A. Introduction to Hopf $*$-superalgebras

In this appendix, we give a detailed list of the definitions and main properties of Hopf superalgebras because this information is scattered in the literature, using various incompatible conventions and notations. Standard references for Hopf algebras and associated structures are [23, 24, 49, 59], and the super case is emphasized in [60]. All vector spaces are over the field $\mathbb{C}$ of complex numbers.

A.1. Hopf superalgebras

A vector space H is a *super vector space* if it can be written as $H = H_0 \oplus H_1$. If an element a is in either H_0 or H_1, we say that it is homogeneous. If $a \in H_0$ (resp. $a \in H_1$), we say that it is even (resp. odd) and its parity is $|a| = 0$ (resp. $|a| = 1$). The concept of super vector space enables us to consider fermions and bosons on the same footing. A super vector space H is a *superalgebra* if it is endowed with an associative product and a unit $1 \in H_0$ and if $|ab| = |a| + |b|$ modulo 2 for any homogeneous elements a and b in H. A superalgebra H is a *Hopf superalgebra* if it is endowed with a coproduct $\Delta : H \longrightarrow H \otimes H$, a counit $\varepsilon : H \longrightarrow \mathbb{C}$ and an antipode $\gamma : H \longrightarrow H$, such that

- Δ is a graded coproduct, i.e. for any homogeneous element a of H, in $\Delta a = \sum a_{(1)} \otimes a_{(2)}$ (using Sweedler's notation) all $a_{(1)}$ and $a_{(2)}$ are homogeneous and $|a| = |a_{(1)}| + |a_{(2)}|$.
- Δ and ε are graded algebra morphisms and γ is a graded algebra anti-morphism, i.e.

$$\Delta(ab) = \sum(-1)^{|a_{(2)}||b_{(1)}|} a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}$$

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$$

$$\gamma(ab) = (-1)^{|a||b|}\gamma(b)\gamma(a);$$

- Δ, ε and γ are unital maps, i.e.

$$\Delta(1) = 1 \otimes 1 \qquad \varepsilon(1) = 1 \qquad \gamma(1) = 1;$$

- Δ is coassociative, i.e.

$$(\Delta \otimes \mathrm{id}) \circ \Delta(a) = (\mathrm{id} \otimes \Delta) \circ \Delta(a) \equiv \Delta^2(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)};$$

- ε is counital, i.e.

$$\sum \varepsilon(a_{(1)})a_{(2)} = \sum a_{(1)}\varepsilon(a_{(2)}) = a;$$

- γ is counital, i.e. $\varepsilon(\gamma(a)) = \varepsilon(a)$, and satisfies the identity

$$\sum \gamma(a_{(1)})a_{(2)} = \sum a_{(1)}\gamma(a_{(2)}) = \varepsilon(a)1.$$

Under these assumptions, the antipode is also a graded coalgebra anti-morphism, that is

$$\Delta\gamma(a) = \sum \gamma(a)_{(1)} \otimes \gamma(a)_{(2)} = \sum (-1)^{|a_{(1)}||a_{(2)}|}\gamma(a_{(2)}) \otimes \gamma(a_{(1)}).$$

A Hopf superalgebra H is *graded commutative* if $ab = (-1)^{|a||b|}ba$, and it is *graded cocommutative* if

$$\Delta a = \sum a_{(1)} \otimes a_{(2)} = \sum (-1)^{|a_{(1)}||a_{(2)}|} a_{(2)} \otimes a_{(1)}$$

for any homogeneous $a, b \in H$. If a Hopf superalgebra H is graded commutative or graded cocommutative, then $\gamma(\gamma(a)) = a$ for any $a \in H$.

A Hopf superalgebra H is graded if there are super vector spaces H_n for $n \in \mathbb{N}$ such that

$$H = \bigoplus_{n\in\mathbb{N}} H_n.$$

If $a \in H_n$ we say that a is a homogeneous element of *degree n*, and we denote the degree of a by $\deg(a)$. Moreover, the degree is an algebra map, i.e. if $a \in H_n$ and $b \in H_m$ then $ab \in H_{n+m}$, and a coalgebra map, i.e. if $a \in H_n$ and $\Delta a = \sum a_{(1)} \otimes a_{(2)}$ then $a_{(1)}$ and $a_{(2)}$ are homogeneous elements and $\deg(a_{(1)}) + \deg(a_{(2)}) = n$. Finally, if a is a homogeneous element, then $\varepsilon(a) = 0$ if $\deg(a) > 0$ and $\varepsilon(1) = 1$.

A.2. Hopf ∗-superalgebra

In quantum field theory, the adjoint operator $a \mapsto a^*$ plays a prominent role. Its existence is one of the basic principles of axiomatic quantum (field) theories [61–63]. It is tied to the definition of a positive quantum field state (i.e. a positive continuous linear functional such that $\rho(1) = 1$ and $\rho(uu^*) \geqslant 0$). Such a ρ allows via the GNS construction the reconstruction of a Hilbert space picture.

Therefore, it is important to specify the interplay between the adjoint operator and the Hopf superalgebra structure. This is done abstractly by defining a star-operation as a bijection $*$: $H \to H$ such that $(a^*)^* = a$ for any $a \in H$, $(\lambda a + \mu b)^* = \bar{\lambda}a^* + \bar{\mu}b^*$ for any a, b in H and any complex numbers λ and μ (with complex conjugate $\bar{\lambda}$ and $\bar{\mu}$). We use De Witt's convention[13]: $(ab)^* = b^*a^*$ because it is compatible with the interpretation of * as the Hermitian adjoint of an operator. The action of the star operation on the tensor product is $(a \otimes b)^* = (-1)^{|a||b|}a^* \otimes b^*$. The compatibility of the star operation with the parity is $|a^*| = |a|$. The compatibility of the star operation with the coproduct is

$$\Delta(a^*) = \sum {a^*}_{(1)} \otimes {a^*}_{(2)} = (\Delta a)^* = \sum (-1)^{|a_{(1)}||a_{(2)}|} {a_{(1)}}^* \otimes {a_{(2)}}^*.$$

[13] In the mathematical literature, one finds also $(ab)^* = (-1)^{|a||b|}b^*a^*$ (e.g. [64, 65]).

The compatibility of the star operation with the counit is $\varepsilon(a^*) = (\varepsilon(a))^*$. The compatibility with the antipode is

$$\gamma(\gamma(a^*)^*) = a.$$

Finally, when the Hopf superalgebra is graded, the star operation must be compatible with this grading: $\deg(a^*) = \deg(a)$.

A.3. The symmetric Hopf superalgebra

If V is a vector space, the symmetric Hopf algebra $S(V)$ was described in pedagogical detail in our paper [15]. Here we consider the symmetric Hopf superalgebra $\mathsf{Sym}(V)$ where V is a super vector space (see [1, appendix 2]). Let $V = V_0 \oplus V_1$ be a super vector space over $\mathbb{C}$, and denote by $|v|$ the parity of a homogeneous element $v \in V$. Let $T(V) = \oplus_{n=0}^{\infty} V^{\otimes n}$ be the tensor algebra on V, with the tensor (free) product $\otimes$, and unit $1 \in \mathbb{C} = V^{\otimes 0}$.

The *symmetric superalgebra on* V is the quotient of $T(V)$ by the supersymmetric (or graded commutative) relation, that is

$$\mathsf{Sym}(V) = T(V)/\langle u \otimes v - (-1)^{|u||v|} v \otimes u\rangle$$

where the elements u, v are homogeneous in V. Since the ideal is generated by a homogeneous relation, the quotient $\mathsf{Sym}(V)$ is still a graded vector space, that is $\mathsf{Sym}(V) = \oplus_{n=0}^{\infty}\mathsf{Sym}^n(V)$, and it has homogeneous components $\mathsf{Sym}^n(V) = \sum_{p+q=n} S^p(V_0) \otimes \Lambda^q(V_1)$, where $S^p(V_0)$ denotes the symmetric p-powers on V_0 and $\Lambda^q(V_1)$ denotes the exterior (skew symmetric) q-powers on V_1. Then, in the quotient $\mathsf{Sym}(V)$ we denote by $\vee$ the concatenation product induced by $\otimes$. Explicitly, for $u_1\vee\cdots\vee u_r \in \mathsf{Sym}^r(V)$ and $v_1\vee\cdots\vee v_s \in \mathsf{Sym}^s(V)$ we have

$$(u_1\vee\cdots\vee u_r)\vee(v_1\vee\cdots\vee v_s) = u_1\vee\cdots\vee u_r\vee v_1\vee\cdots\vee v_s \in \mathsf{Sym}^{r+s}(V).$$

The product $\vee$ is associative (as well as $\otimes$), unital (with unit $1 \in \mathbb{C} = \mathsf{Sym}^0$) and graded commutative, that is $u\vee v = (-1)^{|u||v|} v\vee u$ for homogeneous $u, v \in V$.

As a vector space, $\mathsf{Sym}(V)$ is isomorphic to $S(V_0) \otimes \Lambda(V_1)$, but not as an algebra. In fact, in $S(V_0) \otimes \Lambda(V_1)$ the product is among each tensor component (bosons with bosons, fermions with fermions), while in $\mathsf{Sym}(V)$ we may wish to multiply the two components (bosons with fermions). As an algebra, $\mathsf{Sym}(V)$ is isomorphic to the superalgebra $H_0 \oplus H_1$, where $H_0 = S(V_0) \otimes \oplus_{n=0}^{\infty}\Lambda^{2n}(V_1)$ has even parity and $H_1 = S(V_0) \otimes \oplus_{n=0}^{\infty}\Lambda^{2n+1}(V_1)$ has odd parity. For this, it suffices to check that $H_0\vee H_0$ and $H_1\vee H_1$ are subsets of H_0, and that $H_0\vee H_1$ and $H_1\vee H_0$ are subsets of H_1.

The symmetric superalgebra $\mathsf{Sym}(V)$ can be endowed with a coassociative coproduct $\Delta : \mathsf{Sym}(V) \to \mathsf{Sym}(V) \otimes \mathsf{Sym}(V)$, defined on the generators $v \in V$ as $\Delta v = 1 \otimes v + v \otimes 1$ and extended to products $v_1\vee\cdots\vee v_n$ as an algebra morphism. Due to the graded commutativity of the product $\vee$, the formula for a generic element of length n can be explicitly given in terms of the $(p, n-p)$ shuffles, which are the permutations σ on n elements such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$: if $u = v_1\vee\cdots\vee v_n$,

$$\Delta u = u \otimes 1 + 1 \otimes u + \sum_{p=1}^{n-1} (-1)^F v_{\sigma(1)}\vee\cdots\vee v_{\sigma(p)} \otimes v_{\sigma(p+1)}\vee\cdots\vee v_{\sigma(n)} \qquad (36)$$

where

$$F = \sum_{i=1}^{p} \sum_{j=p+1}^{n} \theta(\sigma(i) - \sigma(j))|v_{\sigma(i)}||v_{\sigma(j)}|$$

with $\theta(\sigma(i)-\sigma(j))=1$ if $\sigma(i)>\sigma(j)$ and $\theta(\sigma(i)-\sigma(j))=0$ if $\sigma(i)<\sigma(j)$. When all operators are odd ($|v_i|=1$ for all i), $(-1)^F$ is the signature $(-1)^\sigma$ of the permutation σ, when all operators are even, $(-1)^F=1$.

The simplest example of equation (36) is

$$\Delta(v_1\vee v_2)=v_1\vee v_2\otimes 1+1\otimes v_1\vee v_2+v_1\otimes v_2+(-1)^{|v_1||v_2|}v_2\otimes v_1.$$

The linear map $\varepsilon:\mathsf{Sym}(V)\to\mathbb{C}$ which is the identity on the scalars $\mathbb{C}=V^0\subset\mathsf{Sym}(V)$, and zero on higher degrees, is a counit for this coproduct. Moreover an antipode is then automatically defined by induction on the length of the elements. In conclusion, the symmetric superalgebra $\mathsf{Sym}(V)$ has the structure of a graded commutative Hopf superalgebra, as defined in appendix A.1.

Appendix B. Cohomology computations for $\mathsf{Sym}(V)$

B.1. Cohomology groups of bosonic $\mathsf{Sym}(V)$

In the bosonic case, i.e. if V is purely even, we can work out the cohomology groups of $\mathsf{Sym}(V)$ as follows.

The symmetric algebra $\mathsf{Sym}(V)$ can be seen as the universal enveloping algebra $\mathsf{U}(V)$ of the Abelian Lie algebra V (with all brackets set to zero). Sweedler proves in [18, theorem 4.1, p 212] that the Hopf algebra cohomology $H^\bullet(\mathsf{Sym}(V))$ is isomorphic to the Hochschild cohomology $HH^\bullet(\mathsf{U}(V))$. This is known to be isomorphic to the Chevalley–Eilenberg cohomology $H^\bullet(V)$ of the Lie algebra. Since V is Abelian, all coboundary operators are zero and $H^\bullet(V)$ is easily computed: $H^\bullet(V)=[\Lambda^\bullet(V)]^*$. Hence $H^1(\mathsf{Sym}(V))=V^*$ and $H^2(\mathsf{Sym}(V))=(\Lambda^2(V))^*=\mathrm{Hom}_{\mathbb{C}}(\Lambda^2(\mathrm{V}),\mathbb{C})$.

B.2. Proofs of cohomological statements

Proof of proposition 3.1. We first show that $T:A_\eta\to A_\chi$ given by $T(a)=\sum\rho(a_{(1)})a_{\underline{(2)}}$ is a comodule map. Using graded cocommutativity we find as required,

$$\begin{aligned}\sum T(a)_{(1)}\otimes T(a)_{\underline{(2)}}&=\sum\rho(a_{(1)})a_{(2)}\otimes a_{\underline{(3)}}\\&=\sum a_{(1)}\otimes\rho(a_{(2)})a_{\underline{(3)}}\\&=\sum a_{(1)}\otimes T(a_{\underline{(2)}}).\end{aligned}$$

We prove secondly that T is a superalgebra isomorphism. We denote the twisted product induced by η by $\circ_\eta$ and the twisted product induced by χ by $\circ_\chi$. Using graded cocommutativity we find as required,

$$\begin{aligned}T(a\circ_\eta b)&=\sum(-1)^{|b_{(1)}||a_{\underline{(2)}}|}\eta(a_{(1)},b_{(1)})T(a_{\underline{(2)}}b_{\underline{(2)}})\\&=\sum(-1)^{|a_{(1)}||a_{(2)}|+|a_{(1)}||a_{(3)}|+|a_{(1)}||a_{\underline{(4)}}|+|a_{(2)}||a_{(3)}|+|a_{(2)}||a_{\underline{(4)}}|+|a_{(3)}||a_{\underline{(4)}}|}\\&\quad\times\partial\rho(a_{(1)},b_{(1)})\chi(a_{(2)},b_{(2)})\rho(a_{(3)}b_{(3)})a_{\underline{(4)}}b_{\underline{(4)}}\\&=\sum(-1)^{|b_{(2)}||a_{\underline{(3)}}|}\rho(a_{(1)})\rho(b_{(1)})\chi(a_{(2)},b_{(2)})a_{\underline{(3)}}b_{\underline{(3)}}\\&=\sum(-1)^{|b_{(1)}||a_{\underline{(2)}}|}\chi(a_{(1)},b_{(1)})T(a_{\underline{(2)}})T(b_{\underline{(2)}})\\&=\sum(-1)^{|T(b)_{(1)}||T(a)_{\underline{(2)}}|}\chi(T(a)_{(1)},T(b)_{(1)})T(a)_{\underline{(2)}}T(b)_{\underline{(2)}})\\&=T(a)\circ_\chi T(b).\end{aligned}$$

For the converse direction we have $A = H$ and the coaction is the coproduct of H. By assumption we have a comodule superalgebra isomorphism $T : A_\eta \to A_\chi$. Let $\rho : H \to \mathbb{C}$ be defined by $\rho := \varepsilon \circ T$. Then ρ is unital, $\rho(1) = 1$. Furthermore, it is invertible with inverse given by $\rho^{-1} = \varepsilon \circ T^{-1}$, and thus it is a 1-cochain on H. Since T is a comodule map we can show using graded cocommutativity

$$\begin{aligned}\sum \rho(a_{(1)})a_{(2)} &= \sum \varepsilon(T(a_{(1)}))a_{(2)} = \sum a_{(1)}\varepsilon(T(a_{(2)}))\\ &= \sum T(a)_{(1)}\varepsilon(T(a)_{(2)}) = T(a).\end{aligned}$$

That is, T is determined by ρ as in the proposition. We show that η and χ are cohomologous through ρ, i.e. $\eta = \partial\rho \star \chi$:

$$\begin{aligned}\eta(a,b) &= \sum (-1)^{|a_{(1)}||a_{(2)}|+|a_{(1)}||a_{(3)}|+|a_{(2)}||a_{(3)}|}\eta(a_{(1)},b_{(1)})\rho(a_{(2)}b_{(2)})\rho^{-1}(a_{(3)}b_{(3)})\\ &= \sum (-1)^{|a_{(1)}||a_{(2)}|+|a_{(1)}||a_{(3)}|+|a_{(2)}||a_{(3)}|}\varepsilon \circ T(\eta(a_{(1)},b_{(1)})a_{(2)}b_{(2)})\rho^{-1}(a_{(3)}b_{(3)})\\ &= \sum (-1)^{|b_{(1)}||a_{(2)}|}\varepsilon \circ T(a_{(1)} \circ_\eta b_{(1)})\rho^{-1}(a_{(2)}b_{(2)})\\ &= \sum (-1)^{|b_{(1)}||a_{(2)}|}\varepsilon(T(a_{(1)}) \circ_\chi T(b_{(1)}))\rho^{-1}(a_{(2)}b_{(2)})\\ &= \sum (-1)^{|b_{(2)}||a_{(3)}|}\rho(a_{(1)})\rho(b_{(1)})\varepsilon(a_{(2)} \circ_\chi b_{(2)})\rho^{-1}(a_{(3)}b_{(3)})\\ &= \sum (-1)^{|b_{(1)}||a_{(2)}|}\partial\rho(a_{(1)},b_{(1)})\chi(a_{(2)},b_{(2)}).\end{aligned}$$

This completes the proof. □

Proof of lemma 3.3. We prove that the map $\mu : Z^1 \times N^1 \to C^1$ given by the convolution product is bijective.

Let $\rho \in C^1$. Define $\zeta : \mathsf{Sym}(V) \to \mathbb{C}$ as follows. Set $\zeta(1) = 1$, set $\zeta(v) = \rho(v)$ for $v \in V$ and extend ζ to all of $\mathsf{Sym}(V)$ as an algebra homomorphism $\zeta(a \vee b) = \zeta(a)\zeta(b)$. (Note that ζ is automatically graded since ρ is graded.) ζ has a convolution inverse $\zeta^{-1}(a) = \zeta(\gamma(a))$ and is thus a 1-cocycle. Now define the 1-cochain $\eta := \zeta^{-1} \star \rho$. Since $\eta(v) = \zeta^{-1}(1)\rho(v) + \zeta^{-1}(v)\rho(1) = \rho(v) - \rho(v) = 0$ for all $v \in V$, η is in N^1. By construction, $\rho = \zeta \star \eta$ and thus μ is surjective.

Now take $\tilde{\zeta} \in Z^1$ and $\tilde{\eta} \in N^1$. Define $\rho := \tilde{\zeta} \star \tilde{\eta}$ and construct as above ζ, η from ρ. Since $\rho(v) = \tilde{\zeta}(1)\tilde{\eta}(v) + \tilde{\zeta}(v)\tilde{\eta}(1) = \tilde{\zeta}(v)$ for $v \in V$ we have by construction of ζ that $\zeta(v) = \tilde{\zeta}(v)$ for $v \in V$. As both ζ and $\tilde{\zeta}$ are algebra homomorphisms they must coincide. Consequently $\eta = \zeta^{-1} \star \rho = \tilde{\zeta}^{-1} \star \tilde{\zeta} \star \tilde{\eta} = \tilde{\eta}$. This shows that μ is injective. □

Proof of lemma 3.4. We show that a 2-cocycle χ on $\mathsf{Sym}(V)$ is a 2-coboundary if and only if it is symmetric, i.e. $\chi(a,b) = (-1)^{|a||b|}\chi(b,a)$ for all a, b in $\mathsf{Sym}(V)$.

Let χ be a 2-coboundary, then there is a 1-cochain ρ such that

$$\chi(a,b) = \partial\rho(a,b) = \sum \rho(a_{(1)})\rho(b_{(1)})\rho^{-1}(a_{(2)} \vee b_{(2)}).$$

Since the symmetric product $\vee$ is graded commutative, the expression on the right is symmetric and $\chi(a,b) = (-1)^{|a||b|}\chi(b,a)$.

For the reciprocal statement, suppose that χ is a symmetric 2-cocycle. We define a 1-cochain on $S(V)$ by induction on $\deg(v)$. Set $\rho(1) = 1$, $\rho(v) = 0$ for all $v \in V$, and $\rho(u \vee v) = \chi^{-1}(u,v)$ for all $u, v \in V$ (which is well defined because χ is symmetric). Now assume that ρ is defined on all elements up to degree $\leqslant n$. For $a, b \in \mathsf{Sym}(V)$ with $\deg(a) + \deg(b) = n + 1$ and $\deg(a), \deg(b) \geqslant 1$ (for a or b of $\deg(0)$ what is to be shown holds automatically), set

$$\rho(a \vee b) = \sum \chi^{-1}(a_{(1)},b_{(1)})\rho(a_{(2)})\rho(b_{(2)}). \tag{37}$$

Then ρ is well defined because χ^{-1} is symmetric, and therefore $\rho(a\vee b) = \rho((-1)^{|a||b|}b\vee a)$, and because $\rho(a\vee(b\vee c)) = \rho((a\vee b)\vee c)$ if $\deg(a) + \deg(b) + \deg(c) = n + 1$ and $\deg(a), \deg(b), \deg(c) \geqslant 1$. To show the latter equality, we use equation (37), the 2-cocycle condition (23) and coassociativity of the coproduct to write

$$\begin{aligned}
\rho(a\vee(b\vee c)) &= \sum(-1)^{|c_{(1)}||b_{(2)}|}\chi^{-1}(a_{(1)}, b_{(1)}\vee c_{(1)})\rho(a_{(2)})\rho(b_{(2)}\vee c_{(2)}) \\
&= \sum(-1)^{|c_{(1)}||b_{(4)}|+|c_{(2)}||b_{(4)}|+|b_{(1)}||b_{(2)}|+|c_{(1)}||c_{(2)}|}\chi^{-1}(a_{(1)}\vee b_{(1)}, c_{(1)}) \\
&\quad\times\chi^{-1}(a_{(2)}, b_{(2)})\chi(b_{(3)}, c_{(2)})\rho(a_{(3)})\rho(b_{(4)}\vee c_{(3)}) \\
&= \sum(-1)^{|b_{(1)}||b_{(2)}|}\chi^{-1}(a_{(1)}\vee b_{(1)}, c_{(1)})\chi^{-1}(a_{(2)}, b_{(2)})\rho(a_{(3)})\rho(b_{(3)})\rho(c_{(2)}) \\
&= \sum(-1)^{|b_{(1)}||b_{(2)}|}\chi^{-1}(a_{(1)}\vee b_{(1)}, c_{(1)})\rho(a_{(2)}\vee b_{(2)})\rho(c_{(2)}) \\
&= \rho((a\vee b)\vee c).
\end{aligned}$$

Finally, inverting (37) to obtain $\rho^{-1}(a\vee b) = \sum\rho^{-1}(a_{(1)})\rho^{-1}(b_{(1)})\chi(a_{(2)}, b_{(2)})$, we can easily show that $\partial\rho = \chi$. In fact

$$\begin{aligned}
\partial\rho(a, b) &= \sum\rho(a_{(1)})\rho(b_{(1)})\rho^{-1}(a_{(2)}\vee b_{(2)}) \\
&= \sum\rho(a_{(1)})\rho(b_{(1)})\rho^{-1}(a_{(2)})\rho^{-1}(b_{(2)})\chi(a_{(3)}, b_{(3)}) \\
&= \sum\varepsilon(a_{(1)})\varepsilon(b_{(1)})\chi(a_{(2)}, b_{(2)}) = \chi(a, b).
\end{aligned}$$

□

Proof of lemma 3.5. We first consider an auxiliary lemma.

Lemma B.1. *Let χ be a 2-cocycle. If χ is symmetric on $V\otimes V$, then χ is symmetric on* $\mathsf{Sym}(V)\otimes\mathsf{Sym}(V)$, *i.e.* $\chi\in Z^2_{\mathrm{sym}}$.

Proof. Let χ be a 2-cocycle such that $\chi(u, v) = (-1)^{|u||v|}\chi(v, u)$. We prove by induction that $\chi\in Z^2_{\mathrm{sym}}$. Suppose that χ is symmetric on all elements $a\otimes b$ with $\deg(a)+\deg(b)\leqslant n$, and let $\deg(a)+\deg(b)+\deg(c) = n+1$, with $\deg(a), \deg(b), \deg(c)\geqslant 1$. Using the graded commutativity of the symmetric product $\vee$, the 2-cocycle condition (23) and the induction hypothesis, we obtain

$$\begin{aligned}
\chi(a, b\vee c) &= \sum(-1)^{|a_{(1)}||a_{(2)}|+|b_{(1)}||b_{(2)}|+|b_{(1)}||b_{(3)}|}\chi^{-1}(b_{(1)}, c_{(1)})\chi(a_{(1)}, b_{(2)})\chi(a_{(2)}\vee b_{(3)}, c_{(2)}) \\
&= \sum(-1)^{|a_{(1)}||a_{(2)}|+|b_{(1)}||b_{(2)}|+|b_{(1)}||b_{(3)}|+|a_{(1)}||b_{(2)}|+|a_{(2)}||b_{(3)}|} \\
&\quad\times\chi^{-1}(b_{(1)}, c_{(1)})\chi(b_{(2)}, a_{(1)})\chi(b_{(3)}\vee a_{(2)}, c_{(2)}) \\
&= \sum(-1)^{|a_{(2)}||b_{(2)}|+|b_{(1)}||b_{(2)}|+|c_{(2)}||c_{(3)}|}\chi^{-1}(b_{(1)}, c_{(1)})\chi(a_{(1)}, c_{(2)})\chi(b_{(2)}, a_{(2)}\vee c_{(3)}) \\
&= \sum(-1)^{|a_{(1)}||c_{(2)}|+|a_{(2)}||b_{(2)}|+|a_{(2)}||c_{(3)}|+|b_{(1)}||b_{(2)}|+|c_{(2)}||c_{(3)}|} \\
&\quad\times\chi^{-1}(b_{(1)}, c_{(1)})\chi(c_{(2)}, a_{(1)})\chi(b_{(2)}, c_{(3)}\vee a_{(2)}) \\
&= (-1)^{|a||b\vee c|}\chi(b\vee c, a).
\end{aligned}$$

This completes the proof. □

We are now ready to prove the main statement that the map $\mu : B^2\times R^2_{\mathrm{asym}}\to Z^2$ given by the convolution product is bijective.

Let χ be a 2-cocycle. Define the Laplace pairing λ by $\lambda(u, v) := \frac{1}{2}(\chi(u, v) - (-1)^{|u||v|}\chi(v, u))$ for all u, v in V extended to $\mathsf{Sym}(V)$ by (1) and (2). λ is an antisymmetric

Laplace pairing according to the definition (29). By lemma 3.2 λ is a 2-cocycle and hence $\sigma := \chi \star \lambda^{-1}$ is also a 2-cocycle. Note that the inverse of λ is the Laplace pairing defined by $\lambda^{-1}(u, v) = -\lambda(u, v)$. Thus, σ evaluated on $V \otimes V$ yields $\sigma(u, v) = \sum(-1)^{|v_{(1)}||u_{(2)}|}\chi(u_{(1)}, v_{(1)})\lambda^{-1}(u_{(2)}, v_{(2)}) = \chi(u, v) - \lambda(u, v) = \frac{1}{2}(\chi(u, v) + (-1)^{|u||v|}\chi(v, u))$. That is, σ is symmetric on $V \otimes V$. By lemma B.1 this implies that σ is symmetric on all of $\mathsf{Sym}(V)$ and thus by lemma 3.4 a 2-coboundary. By construction $\chi = \sigma \star \lambda$, i.e. χ can be written as a product of a 2-coboundary σ and an antisymmetric Laplace pairing λ. Hence μ is surjective.

Now take $\tilde{\sigma} \in B^2$ and $\tilde{\lambda} \in R^2_{\text{asym}}$. Define the 2-cocycle $\chi := \tilde{\sigma} \star \tilde{\lambda}$. Construct $\sigma \in B^2$ and $\lambda \in R^2_{\text{asym}}$ out of χ as above. Then for u, v in V, $\chi(u, v) = \sum(-1)^{|v_{(1)}||u_{(2)}|}\tilde{\sigma}(u_{(1)}, v_{(1)})\tilde{\lambda}(u_{(2)}, v_{(2)}) = \tilde{\sigma}(u, v) + \tilde{\lambda}(u, v)$. Hence $\lambda(u, v) = \frac{1}{2}(\chi(u, v) - (-1)^{|u||v|}\chi(v, u)) = \frac{1}{2}(\tilde{\sigma}(u, v) + \tilde{\lambda}(u, v) - (-1)^{|u||v|}(\tilde{\sigma}(v, u) + \tilde{\lambda}(v, u))) = \tilde{\lambda}(u, v)$. We have used that $\tilde{\lambda}$ is antisymmetric by assumption while $\tilde{\sigma}$ is symmetric by lemma 3.4. Since λ and $\tilde{\lambda}$ are Laplace pairings coinciding on $V \otimes V$ they must be identical. Consequently $\sigma = \chi \star \tilde{\lambda}^{-1} = \tilde{\sigma} \star \tilde{\lambda} \star \tilde{\lambda}^{-1} = \tilde{\sigma}$. This shows that μ is injective. □

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